



معادلات تفاضلية (١)

د.تقي الشطناوي

للطالبة المبدعة
سباً جهاد

إرادة - ثقة - تغيير

- Differential Equation $\rightarrow$ متغير constant $*, +, -$

- Defn: A D.E is an equation contains at least one derivative

Ex: I] $y' = 2x + 3$

II] $2xy'' + 3y' + 2y = \sin x$

$- y' = \frac{dy}{dx}$ y : dep. var

x : indep. var

$- y'' = \frac{d^2y}{dx^2}$, $y''' = y^{(3)} = \frac{d^3y}{dx^3}$

$\Rightarrow$ Notes: $y^{(n)}$: derivative of order (n)

y^n : power

* There are two types of D.E

I] ordinary D.E (O.D.E)

if there are one indep. var $[y, \frac{dy}{dx}, \dots, y^{(n)}]$

Ex: $y' + \sin xy = 3x$

II] partial D.E (P.D.E)

the number of indep. variable at least two

$u(x, t)$

$\frac{du}{dx}, \frac{du}{dt}, \frac{d^2u}{dx^2}, \frac{d^2u}{dt^2}$ EX: $u_t + u_{xx} = 0$
 u_x, u_t, u_{xx}, u_{xt}

$\Rightarrow$ Order of D.E "is the highest derivative"

سأكتبه

Ex: Find the order of D.E order

I] $(\cos x)y = x^2y'' - y'$ order 3

II] $(y'')^3 - ty' = t^2$ order 2

III] $(y''')^{(3)} - ty' = t^2$ order 5 $\rightarrow 2+3$

IV] $y^{(10)} [y + y^{(11)}]'' = y^{(14)}$ order 13

$y'' + y^{(13)}$

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ترتيب المعادلة

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- Defn: Linear D.E dep y, y', y'', y'''

O.D.E of order (n) is called linear if it has the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$$

$$(y-y'') / (y'-y'') \rightarrow \text{non linear}$$

$$y = e^x \quad e^x = \sum \frac{x^n}{n!}$$

Ex: classify the following D.E in terms of - order

- linear, non linear - P.D.E, O.D.E

$$[1] xy''' + 2x^2y'' - 3y + 9x^2$$

order: 3, Linear, O.D.E

$$[2] xx' - tx'' = t^3$$

order 2, non linear, O.D.E " Ordinary

$$[3] \phi''' = 9r\phi - e^{\phi^{(5)}}$$

order 5, non linear [exponential function]

$$\sum_{n=1}^{\infty} \frac{(\phi)^{(5)}}{n!} \rightarrow \frac{\phi^{(5)}}{1!} = \frac{(\phi^{(5)})^2}{2!}, \text{ non linear}$$

Defn: A function $y = P(x)$

is a solution to ODE if it satisfies the

Ex: let $y'' = \frac{2}{x^2} y = 0$ show that $P(x) = x^2 - x^{-1}$ is solve

$$y' = 2x + x^{-2}, \quad y'' = 2 - 2x^{-3}$$

$$(2 - 2x^{-3}) = \frac{2}{x^2} (x^2 - x^{-1})$$

$$2 - 2x^{-3} = 2 + \frac{2x^{-1}}{x^2}$$

$$-2x^{-3} + 2x^{-3} = 0$$

* $y = f(x)$ Explicit $\rightarrow$ مشتق و جواب

* Implicit $\ln y = y + x^2$ $\rightarrow$ مشتق و جواب $\rightarrow$ مشتق و جواب $\rightarrow$ مشتق و جواب

Ex: show that $y = \ln y + x^2 + 1$ is an implicit soln for.

121 $(y-1)y' = 2xy$

* اندي شكل الحالة 13 اشتقاق

use implicit $y = \ln y + x^2 + 1$

الحالة 2

derivative: $\frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} + 2x \rightarrow (y' - \frac{1}{y} y' = 2x) * y$

$y'y - y' = 2xy$

$y'(y-1) = 2xy \checkmark$

Ex: IF $y = e^{-3}$ is a soln for $y'' + xy' + p(x)y = 0$ Find $p(x)$

$y' = -3e^{-3}$

$9e^{-3} + -3e^{-3}x + p(x)e^{-3} = 0$

$y'' = 9e^{-3}$

$e^{-3} (9 - 3x + p(x)) = 0$

$e^{-3} \neq 0 \quad \downarrow = 0 \quad p(x) = 3x - 9$

Ex: IF $y = t^n$ is soln for $t^2 y'' + 6ty' + 4y = 0 \quad t > 0$

Find $n \quad y' = n t^{(n-1)}$

$t^2 (n(n-1)t^{(n-2)} + 6t(n t^{(n-1)}) + 4t^n$

$y'' = n(n-1)t^{(n-2)}$

$t^n (n(n-1) + 6n + 4)$

$t^n \neq 0, \quad n(n-1) + 6n + 4 = 0$

$n^2 + 5n + 4, \quad (n+1)(n+4), \quad n = -1 \quad n = -4, \quad y_1 = t^{-1}, \quad y_2 = t^{-4}$

Ex: How show that $y^2 + (x^2 + 1)y - 3x^3 - 6 = 0$ is soln for

$2xy^2 - 9x^2y + (2y^2 + x^2y + y)y'$

$\frac{dy}{dx} (2y^2 + 2yx + x^2y + y) - 9x^2 = 0 \quad \times y$

$y'(2y^2 + x^2y + y) + 2xy^2 - 9x^2y \checkmark$

* Initial value problem IVP

$$IVP \neq D.E + IC \Rightarrow IVP$$

Defn is ODE with initial conditions (IC's) general form is

$$F(y, y', \dots, y^{(n)}) = 0$$

with IC

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \dots$$

where x_0 fixed

$$y_0, y_1, y_{n-1} \text{ are constant}$$

IC 3 ليا order 3 ان order 3 يكون بعد ال

Note: use IC to find the constant in general soln

Ex: If $y_{(x)} = C_1 e^{-x} + C_2 e^{2x}$ is a soln to $y'' - y' - 2y = 0$

, $y(0) = 2$, $x_0 = 0$ Find C_1 and C_2

$$y(0) = -3 \quad y_0 = 2, y_1 = -3$$

$$y(0) = C_1 e^{-0} + C_2 e^{2 \cdot 0} = 2 \quad \text{--- (1) + (2)}$$

$$C_1 + C_2 = 2 \quad \text{--- (1)} \quad C_2 = \frac{1}{3}$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} \quad C_1 = \frac{7}{3}$$

$$-3 = -C_1 + 2C_2 \quad \text{--- (2)}$$

Ex: If $4x^2 - y^2 = C$ is soln for $yy' - 4x = 0$, $y(1) = 3$ Find C

$$x = 1 \quad y = 3 \quad 4 - 9 = C \quad C = -5$$

* Existence and unique consider IVP

$y' = F(x, y)$ $y(x_0) = y_0$ if $F(x, y)$ and $\frac{dF}{dy}$ are const function

on $a \leq x \leq b$, $c \leq y \leq d$ contains (x_0, y_0) then IVP has a unique soln

بعت y في حيزه والفرق الثاني او $F(x, y)$ متناهي

يكونه لازم يكون $F(x, y)$ متناهي

y لازم متناهي اشتغلي بنسبة الى x ثابت

$$\frac{dF}{dy}$$

Subject : Ch "1"

Ex: Does IVP $3y' - xy^3 = x^2, y(1) = 3$ has Unique soln

$$3y' = xy^3 + x^2$$

$$y' = \frac{xy^3}{3} + \frac{x^2}{3} \rightarrow \text{cont} \Rightarrow \text{polynome}$$

$$f(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{4}{3} \times 3xy^2 = xy^2 \rightarrow \text{conts} \rightarrow \text{d.p.s}$$

so has Unique solution

Ex: Find the largest interval where IVP has Unique soln

$$y' = \frac{y}{(x+1)(x-2)(x+3)}$$

$$y(0) = 3$$

بجانب الخيارات

Largest interval

بجانب الخيارات $\frac{\partial f}{\partial y}$
 نختار الفترة التي تحتوي على x_0

$$y' = \frac{y}{(x+1)(x-2)(x+3)}$$

$$\rightarrow \text{cont } R = \mathbb{R} \setminus \{-1, 2, -3\}$$

Interval of cont $f(x, y)$

$$(-\infty, -3) \cup (-3, -1) \cup (-1, 2) \cup (2, \infty)$$

$$\frac{\partial f}{\partial y} = \frac{1}{(x+1)(x-2)(x+3)}$$

Interval of cont $\frac{\partial f}{\partial y}$ is cont

$$\text{For } R \setminus \{x = -1, 2, -3\}$$

$$(-\infty, -3) \cup (-3, -1) \cup (-1, 2) \cup (2, \infty)$$

$$x = 0$$

As $x_0 \in (-1, 2)$ largest interval for Unique soln is $(-1, 2)$

Ex: $y' + (\tan x)y = \sin x, y(\pi) = 0$

$$y' = \sin x - (\tan x)y \rightarrow \tan x = \frac{\sin x}{\cos x}, \cos x = 0 \text{ not cont}$$

$f(x, y)$ cont on $\mathbb{R} \setminus \{\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}\}$

Interval of cont

$$\frac{\partial f}{\partial y} = -\tan x$$

$$(-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$$

$$\text{As } x_0 = \pi \in (\frac{\pi}{2}, \frac{3\pi}{2})$$

S I G M A N O T E B O O K

* First order D.E

III Seperable Equation IF ODE can be written as

$f(x) dx = g(y) dy$ Then is called seperable equation.

$$y' = \frac{dy}{dx}$$

Method: Integrate both sides

$$\int f(x) dx = \int g(y) dy + C$$

Ex: Solve // Find the general soln

$$\text{III } y' = \frac{x-5}{y^2} \rightarrow \frac{dy}{dx} = \frac{x-5}{y^2} \rightarrow \int y^2 dy = \int (x-5) dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$$y = \sqrt[3]{3\left(\frac{x^2}{2} - 5x + C\right)}$$

IC $\rightarrow$ to Find C

$$\text{12) } y' = (1+y^2) \tan x \quad y(0) = \sqrt{3}$$

$$\frac{dy}{dx} = (1+y^2) \tan x$$

$$\int \frac{1}{1+y^2} dy = \int \tan x dx$$

$$\tan^{-1} y = -\ln |\cos x| + C$$

use IC $y(0) = \sqrt{3}$ to Find C $x=0 \quad y = \sqrt{3}$

$$C = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\tan^{-1} y = -\ln |\cos x| + \frac{\pi}{3}$$

$$y = \tan(-\ln |\sec x| + \frac{\pi}{3})$$

$$\text{13) } y' = y^2 + 4$$

$$\frac{1}{y^2+4} dy = dx$$

$$\rightarrow \int dx = \int \frac{1}{y^2+4} dy = \int \frac{1}{y-2} + \frac{1}{y+2} dx$$

$$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$1 = A(y+2) + B(y-2)$$

$$y=2 \rightarrow A = \frac{1}{4} \quad y=-2 \rightarrow B = -\frac{1}{4}$$

$$\frac{1}{4} \ln |y-2| - \frac{1}{4} \ln |y+2| = x + C$$

$$\frac{1}{4} \ln \left| \frac{y-2}{y+2} \right| = x + C$$

$$K = 4C$$

$$\ln \left| \frac{y-2}{y+2} \right| = 4x + K$$

$$y-2 = (y+2) e^{4x+K}$$

12] Linear Equation

- Standard Form

$$y' + p(x)y = Q(x)$$

is called linear in y
 dep var

$$y' + 2xy = e^x$$

$$p(x) = 2x$$

$$Q(x) = e^x$$

- معادلة y' خطية -

- معادلة y' خطية في y
 y متغير المعتمد

in x only $p(x)$

- Method: Find the integrating factor $M(x) = e^{\int p(x) dx}$

$$M'(x) = p(x) \cdot e^{\int p(x) dx}$$

(E.g)' $\rightarrow$ مشتق

$$M' = M \cdot p(x)$$

$$E.g' + y p'$$

$$p(x) = \frac{M'(x)}{M(x)}$$

$$y' + p(x)y = Q(x)$$

Multiply D.E by I.F = $M(x)$

$$M(x)y' + \underbrace{M(x)p(x)}_{M'(x)}y = M(x)Q(x)$$

$$M(x)y' + M'(x)y = M(x)Q(x)$$

$$\int (M(x)y)' = \int M(x)Q(x)$$

$$M(x)y = \int M(x)Q(x) dx + C$$

$$y = \frac{1}{M(x)} \left[\int M(x)Q(x) dx + C \right]$$

Ex 1: Solve $x y' + y = \sin x$, $x \neq 0$

II $I = y' dx$

$$\div x \Rightarrow y' + \frac{1}{x} y = \frac{\sin x}{x}$$

- linearizing, $p(x) = \frac{1}{x}$, $Q(x) = \frac{\sin x}{x}$ انتبه في الاشارة اذا سالنا موجب

$$M(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Solution

$$y(x) = \frac{1}{x} \left[\int x \cdot \frac{\sin x}{x} dx + C \right]$$

$$= -\frac{1}{x} (\cos x + C)$$

121 Solve $t y' - 2y = 5t^2$ $y(1) = 2$ $t > 0$

$$\div t = y' - \frac{2}{t} y = 5t$$

Linearizing, with $p(t) = -\frac{2}{t}$, $Q(t) = 5t$

$$M(x) = e^{\int -\frac{2}{t} dt} = e^{-2 \ln t} = e^{\ln t^{-2}} = t^{-2} = \frac{1}{t^2}$$

من قوانين ln

$$\text{solu } y(t) = t^2 \left[\int \frac{1}{t^2} \cdot 5t dt + C \right]$$

$$y(t) = t^2 [5 \ln t + C]$$

Use IC $y(1) = 2$ to Find C

$$\text{at } t = 1 \quad y(1) = 2$$

$$2 = (1)^2 [5 \ln(1) + C] \rightarrow C = 2$$

$$\text{general solu } y(t) = t^2 [5 \ln t + 2]$$

* اذا اعطاك سؤال IC $y(x) = B$ لازم ما اذني السعاري بدلان "C"

$$* M(x) = e^{\int p(x) dx}$$

$$M'(x) = p(x) M(x) \rightarrow p(x) = \frac{M'(x)}{M(x)}$$

131 If $M(x) = 3x^2$ in an I.F. For $2y' + x f(x)y = 3x - 4$

Find $f(x)$

II $\rightarrow$ $y' + \frac{x}{2} f(x)y = \frac{3}{2}x - 2$ (اشق اول)

linearizing, $p(x) = \frac{x}{2} f(x)$ $Q(x) = \frac{3}{2}x - 2$

$$p(x) = \frac{M'(x)}{M(x)} = \frac{6x}{3x^2} = \frac{2}{x}$$

$$p(x) = \frac{x}{2} \cdot \frac{2}{x} = 1$$

$$f(x) = \frac{2}{x} \times \frac{2}{x} = \frac{4}{x^2}$$

121 ثاني طريقه

$$M(x) = e^{\int p(x) dx} = 3x^2 = e^{\int p(x) dx}$$

$$\text{Take } \ln = (\ln 3x^2)' = (\int p(x) dx)' \rightarrow \text{اشق}$$

$$\frac{6x}{3x^2} = p(x)$$

$$f(x) = \frac{2}{x} p(x) = \frac{4}{x^2}$$

141 solve $(x + 4y^2) dy + 2y dx = 0$

linear y مشي y^2 $\rightarrow$ linear y مشي y^2

separate مشي

$$* \text{ linear in } x : ((x + 4y^2) dy + 2y dx) \div dy = x + 4y^2 + 2yx' = 0$$

$$y \text{ مشي } \rightarrow x' + \frac{2}{y}x = -4y \rightarrow \left[\frac{x}{\frac{2}{y}} + \frac{2yx'}{2y} = \frac{-4y^2}{2y} \right]$$

$$M(y) = e^{\int \frac{2}{y} dy} = y^2$$

$$x(y) = \frac{1}{M(y)} \left[\int Q(y) \cdot M(y) dy + C \right]$$

$$\frac{1}{y^2} \left[\int -2y \cdot \sqrt{y} dy + C \right]$$

[2.6] Bernoulli Equation

مشي y^n $\rightarrow$ linear مشي y^n

$$y' + p(x)y = Q(x)y^n \quad \text{if } n=0 \rightarrow \text{linear}$$

$$\text{if } n=1 \rightarrow y' + p(x)y = Q(x)y \quad y' = \frac{dy}{dx}$$

$$dy = y(-p(x) + Q(x)) dx = \frac{1}{y} dy = (Q(x) - p(x)) dx \text{ separate}$$

Note: $y' + p(x)y = Q(x)y^n$ where $n \neq 0, 1$ is

Bernoulli Equation $n \in \mathbb{R} - \{0, 1\}$

* Method:

consider $V = y^{(1-n)}$

Transform Bernoulli to linear in V

$$\div by y^n [y' + p(x)y = Q(x)y^n]$$

$$= y' y^{-n} + p(x) y^{1-n} = Q(x)$$

$$V = y^{1-n} \quad V' = (1-n) y^{-n} y' \rightarrow$$

$$y' = \frac{V'}{(1-n)y^{-n}}$$

$$\left(\frac{V'}{(1-n)y^{-n}} y^{-n} + p(x)V = Q(x) \right) * (1-n)$$

linear in V

$$M(x) = e^{\int (1-n)p(x) dx}$$

$$V(x) =$$

* To Find solutioning use $V = y^{(1-n)}$

use TC

$y =$

ياخد قيمة TC للفرد الجديد

الرجعي للفرد ثم جدي "C"

Ex: Solve $y' - 5y = -\frac{5}{2} x y^3$

II) $V = y^{-2}$ III) linear in $V \rightarrow V' + (1-n)p(x)V = (1-n)Q(x)$

$$V' + 10V = 15x$$

$$M(x) = e^{\int 10 dx} = e^{10x}$$

$$V(x) = \frac{1}{e^{10x}} \left[\int 15x e^{10x} dx + C \right]$$

$$V(x) = \frac{1}{e^{10x}} \left[\frac{15x e^{10x}}{10} - \frac{1}{20} e^{10x} + C \right]$$

$$V(x) = \frac{1}{2} x - \frac{1}{20} + C e^{-10x}$$

$$y^{-2} =$$

$$(y^{-2})^{-\frac{1}{2}} = \left(\frac{1}{2} x - \frac{1}{20} + C e^{-10x} \right)^{-\frac{1}{2}}$$

$$y =$$

$$\sqrt{\frac{1}{2} x - \frac{1}{20} + C e^{-10x}}$$

تكاملك كسور جزئي

$$\begin{array}{r|l} 5x & e^{10x} \\ \hline & 5x e^{10x} \\ & 5 e^{10x} \\ & 0 \end{array}$$

Ex: Solve $(x-2)y' - 5(x-2)^2 y^{\frac{1}{2}} = y$

"C" value

① $y' - \frac{1}{(x-2)} y = 5(x-2) y^{\frac{1}{2}}$

$y(3) = 4$

$V = y^{1-n} \rightarrow y^{\frac{1}{2}}$

$V' + (1-n) p(x)V = (1-n) q(x)$

$V' + \frac{-1}{2(x-2)} y = \frac{1}{2} 5(x-2)$

$M(x) = e^{\int \frac{-1}{2(x-2)} dx} = (x-2)^{-\frac{1}{2}}$

$V(x) = (x-2)^{\frac{1}{2}} \cdot \left[\int (x-2)^{-\frac{1}{2}} \cdot \frac{5}{2} (x-2) dx \right]$

$V(x) = (x-2)^{\frac{1}{2}} \cdot \left[\frac{5}{2} \int (x-2)^{\frac{1}{2}} dx \right]$

$V(x) = (x-2)^{\frac{1}{2}} \cdot \left(\frac{5}{2} \cdot \frac{2}{3} (x-2)^{\frac{3}{2}} + C \right)$

$y^{\frac{1}{2}} = (x-2)^{\frac{1}{2}} \cdot \left(\frac{5}{3} (x-2)^{\frac{3}{2}} + C \right)$

$2 = \frac{5}{3} + C \rightarrow C = \frac{1}{3}$

* 2.4 Exact Equation

معنى (x, y) الاشتقاق

II Form $M(x, y) + N(x, y) y' = 0$

يمكن أن تكون نسبة للمتغير

III Form $M(x, y) dx + N(x, y) dy = 0$

واحد للمتغير ثابت

if $\Rightarrow$ D.E is exact

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

بمعنى اشتقاق ودرجات متساوية يعني exact

Ex: III which of the following D.E is exact

$(\cos x \cos y + 2x) dx - (\sin x \sin y + 2y) dy = 0$

مشتق y مع نسبة dx و مشتق x مع نسبة dy

$\frac{\partial M}{\partial y} = -\cos x \sin y$, $\frac{\partial N}{\partial x} = -\sin y \cos x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

121 $(2xy + \pi) = (x^2 - 1)y'$

① → Form $M dx + N dy = 0$

$(2xy + \pi) - (x^2 - 1)y' = 0$ ✓
 $\underbrace{\hspace{1.5cm}}_M \underbrace{\hspace{1.5cm}}_N$

$\frac{\partial M}{\partial y} = 2x$ $\frac{\partial N}{\partial x} = -2x \rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Not exact

* أنواع الثاني من الأساليب يبحثون exact ويطلبون جواب ؟

Ex: Find $N(x,y)$ such that the D.E is exact

$(x^{-1/2} y^{1/2} + x(x^2 + y)^{-1}) dx + N(x,y) dy = 0$
 $\underbrace{\hspace{2cm}}_M$

As it is exact $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = x^{-1/2} \cdot \frac{1}{2} y^{-1/2} + x(-1)(x^2 + y)^{-2} \Rightarrow$ اشتقبت بنسبة إلى y

$\int \frac{\partial N}{\partial x} dx = N(x,y)$

$\int \frac{1}{2} x^{-1/2} y^{-1/2} + \int -x(x^2 + y)^{-2}$ ← x بنسبة إلى x

$\frac{1}{2} y^{-1/2} \int x^{-1/2} dx - \int x(x^2 + y)^{-2}$

$y^{-1/2} x^{1/2} - \int x(x^2 + y)^{-2}$ $u = x^2 + y \quad du = 2x$

$y^{-1/2} x^{1/2} - \int x u^{-2} \frac{du}{2x}$

$N(x,y) = y^{-1/2} x^{1/2} + \frac{1}{2} (x^2 + y)^{-1} + g(y)$ →

الاشتقاق مع x with respect to x نتيجته x بنسبة إلى x بنسبة إلى x

$\int M(x,y) dy = - + h(x)$ $\frac{dx}{dy}$ بنسبة إلى dy بنسبة إلى dy

Ex: Find K so that D.E is exact $(6xy^3 + \cos y) dx + (Kx^2y^2 - x \sin y) dy = 0$
 $\underbrace{\hspace{1.5cm}}_M \underbrace{\hspace{1.5cm}}_N$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
 $= 18xy^2 - \sin y = 2Ky^2x - \sin y$

$2K = 18 \Rightarrow K = 9$

Ex: Find a and b so that the D.E is exact $(ax^2 - by^2) dx - (x^3 + 2xy) dy = 0$
 $\underbrace{\hspace{1.5cm}}_M \underbrace{\hspace{1.5cm}}_N$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow ax^2 - 2by = -3x^2 - 2x$
 $\underbrace{\hspace{1.5cm}}_M \underbrace{\hspace{1.5cm}}_N$ $a = -3$

$-2b = -2 \rightarrow b = 1$

$$\text{Ex: } -M(x,y) + (xe^{xy} + \frac{1}{x})dy = 0$$

$$\frac{dM}{dy} = \frac{dN}{dx}$$

$$\frac{dN}{dx} = [xye^{xy} + e^{xy} + \frac{-1}{x^2}]$$

y dā fāsi dāb

$$\int xye^{xy} + \int e^{xy} + \int -\frac{1}{x^2}$$

$$u = xy \quad du = x \quad dx$$

$$du = e^{xy} \rightarrow \frac{e^{xy}}{x}$$

$$xy \frac{e^{xy}}{x} = \int \frac{e^{xy}}{x} \cdot x$$

$$(ye^{xy} - \frac{e^x}{x})$$

$$\frac{e^{xy}}{x} + -\frac{1}{x^2} y$$

$$ye^{xy} - \frac{e^{xy}}{x} + \frac{e^{xy}}{x} = \frac{1}{x^2} y$$

$$M(x,y) = ye^{xy} - \frac{1}{x^2} y + C$$

Method to solve exact Eqn.

$$\text{Let } M(x,y)dx + N(x,y)dy = 0$$

$$\frac{dM}{dy} = \frac{dN}{dx}$$

Steps:

[1] Let solution for exact is $F(x,y) = C, C \in \mathbb{R}$

[2] to Find $F(x,y)$ we can use $\frac{\partial F}{\partial x} = M(x,y)$ or $\frac{\partial F}{\partial y} = N(x,y)$

* M قابل دالة $F(x,y)$ line

or $N(x,y)$

[3] $F(x,y) = \int M(x,y) dx + g(y)$

* مشتقات استخدام $M(x,y)$

[4] use $\frac{\partial F}{\partial y} = N$ to Find $g'(y)$

* عند إيجاد $g(y)$ باستخدام $N(x,y)$

[5] to find $g'(y) \rightarrow \int g'(y) dy = g(y)$

والدخول صحيح

$$\text{Ex solve } (\cos x \sin x - xy^2)dx + y(1-x^2)dy = 0$$

$\underbrace{\hspace{1cm}}_M \quad \underbrace{\hspace{1cm}}_N$

تفقدنا
exact

$$0 - 2xy = -2xy \checkmark$$

[1] $F(x,y) = C$

$$\int \frac{\partial F}{\partial x} = \int M(x,y) \rightarrow F(x,y) = \int M(x,y) dx$$

$$\text{substitue } \int \cos x \sin x - xy^2 dx + g(y)$$

$$\int \frac{1}{2} \sin 2x - xy^2 dx$$

$$F(x,y) = -\frac{1}{2} \cos(2x) - \frac{y^2 x^2}{2} + g(y)$$

$$\frac{\partial F}{\partial y} = N(x,y)$$

$$0 = -\frac{1}{2} x^2 y \cdot 2 + g'(y) = y(1-x^2)$$

ثابت x

$$g'(y) = y - yx^2 + x^2 y \rightarrow$$

$$g'(y) = y$$

$$\int g'(y) = \frac{y^2}{2} + g(y)$$

هنا الخطأ في

افتقارنا إذا

طبي، واحد ملا

في خطا

- solu

$$F(x,y) = C$$

$$-\frac{1}{4} \cos(2x) - \frac{1}{2} x^2 y^2 + \frac{y^2}{2} = C$$

Ex: solve $(1 + e^x y + x e^x y) dx + (x e^x + 2) dy = 0$ $y(0) = 1$

$$M_y = e^x + x e^x$$

$$N_x = x e^x + e^x$$

is exact $M_y = N_x$

$$(1) \frac{\partial F}{\partial y} = N = x e^x + 2$$

$$(2) F(x, y) = \int (x e^x + 2) dy = (x e^x + 2)y + h(x)$$

$$\frac{\partial F}{\partial x} = M \rightarrow h'(x) + x y e^x + e^x y = 1 + e^x y + x e^x y$$

$$h'(x) = 1 \rightarrow h(x) = x$$

∴ general soln $F(x, y) = C$

$$(x e^x + 2)y + x = C$$

$$y(0) = 1 \quad (0 + 2) + 0 = C = 2$$

-2.5 special Integrating Factor if D.E is

$$M(x, y) dx + N(x, y) dy = 0 \quad M_y \neq N_x \quad \text{not exact}$$

if $M(x, y)$ is Function such that $\underbrace{M(x, y)}_{M^*} dx + \underbrace{N(x, y)}_{N^*} dy = 0$

$$\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

then $M(x, y)$ special I.F

Ex: if $(12 + 5xy) + (6xy^{-1} + 3x^2) dy = 0$ has I.F $M(x, y) = x^n y^n$

Find n $x^n y^n (12 + 5xy) dx + x^n y^n (6xy^{-1} + 3x^2) dy = 0$

$$(12x^n y^n + 5x^{n+1} y^{n+1}) dx + (6x^{n+1} y^{n-1} + 3x^{n+2} y^{n+1}) dy = 0$$

$$\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$$24x^n y^{n-1} + 15x^{n+1} y^{n-1} = 6(n+1)x^{n+1} y^{n-1} + 3(2+n)x^{n+2} y^{n-1}$$

$$24 = 6n + 6 \rightarrow n = 3$$

Note: How to Find special I.F

كيف نجد I.F. بالطريقة

[1] case: if $M_y - N_x = \dots$

$M(x)$ المعامل الذي استعملنا

depend only on x Then $M(x) = e^{\int \frac{M_y - N_x}{N} dx}$

عشان انزل الـ exact

انتبه زغيره كتابهم

[2] case: if $N_x - M_y = \dots$

depend only on y Then $M(x) = e^{\int \frac{N_x - M_y}{M} dy}$

Ex: if $M(x) = y$ is I.F for $M dx - N dy = 0$ Find $N_x - M_y$

$$M(x) = e^{\int \frac{N_x - M_y}{M} dy}$$

تخلص من e في

$$1 = \int \frac{N_x - M_y}{M} dy$$

تخلص من $N_x - M_y$ في مشتقة

$$\frac{1}{y} = \frac{N_x - M_y}{M}$$

$$\rightarrow N_x - M_y = \frac{M}{y}$$

Ex: Find I.F for D.E $(2x^2y + xe^x) dx + (x^3 + xy^3) dy = 0$

that trans Form D.E is not exact

$$\frac{\partial M}{\partial y} = 2x^2 \neq \frac{\partial N}{\partial x} = 3x^2 + y^3$$

$$\frac{M_y - N_x}{N} = \frac{2x^2 - 3x^2 - y^3}{x^3 + xy^3} = \frac{-x^2 - y^3}{x^3 + xy^3} = \frac{-(x^2 + y^3)}{x(x^2 + y^3)} = -\frac{1}{x}$$

$$M(x) = e^{\int -\frac{1}{x} dx} = x^{-1}$$

$$\frac{1}{x} (2x^2y + xe^x) dx + \frac{1}{x} (x^3 + xy^3) dy = 0$$

$$\frac{\partial M^*}{\partial y} = 2x = \frac{\partial N^*}{\partial x} = 2x \therefore \text{exact} \checkmark$$

Ex: solve $y dx + (2x - ye^y) dy = 0$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2 \rightarrow \text{I.F. } \frac{M(x) - N_x}{N} = \frac{1 - 2}{2x - ye^y} \text{ Not}$$

$$\frac{N(x) \cdot M(x)}{m} = \frac{2x - 1}{y} = \frac{1}{y} \therefore \text{I.F.}$$

$$\text{I.F } M(x) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

- Multiply D.F by $M(x) = y$

$$y^2 dx + (2xy - y^2 e^y) dy$$

$$2y = 2y \quad \checkmark$$

$$F(x, y) = \int y^2 dx = y^2 x + g(y)$$

$$\frac{\partial F}{\partial y} = 2xy + g'(y) = 2xy - y^2 e^y$$

$$g'(y) = -y^2 e^y$$

$$g(y) = -\int y^2 e^y$$

$$g(y) = -e^y y^2 + 2y e^y - 2e^y$$

$$C = y^2 x - e^y (y^2 - 2y + 2)$$

$$\begin{array}{r|l} y^2 & e^y \\ \hline & y^2 e^y \\ & 2y e^y \\ & 2 e^y \\ & 0 \end{array}$$

$$\text{Ex 6: IF } M(x, y) = N(x, y)$$

Find the general solen to D.E $\frac{M}{x} dx - \frac{N}{y} dy = 0$

$$\frac{M(x, y) - N(x, y)}{N(x, y)} = x = \frac{M}{N} = x + 1 \rightarrow M = N(x+1)$$

$$N \div \frac{N(x+1) dx - \frac{N}{y} dy = 0 \Rightarrow \frac{(x+1) dx}{x} - \frac{1}{y} dy$$

$$\int \frac{(x+1) dx}{x} = \int \frac{1}{y} dy$$

$$x + \ln x = \ln y$$

$$y = e^{x + \ln x}$$

2.6 :- Homogeneous

~~Defn~~ Defn:- IF D.E $y' = F(x, y)$ can be written as $y' = F(\frac{x}{y})$

or $y' = F(\frac{y}{x})$ Then, called Homog

Ex: $2\frac{x}{y} + 10\frac{y}{x} + 5 = 2\frac{x}{y} + 10(\frac{y}{x})^{-1} + 5$

* كل دالة y' متجانسة

تكون إما $\frac{x}{y}$ أو $\frac{y}{x}$

عشان افهم

Ex: is the following D.E is Homog

$y' = \frac{y-4x}{x-y}$ $\frac{y(1-\frac{4x}{y})}{y(\frac{x}{y}-1)}$ $F(\frac{x}{y})$ is Homog

[2] $y' = \frac{x}{y} + \ln x$ Not Homog

عشان ln

[3] $y' = \frac{y}{x} e^{\frac{x}{y}} + (\frac{x}{y})^{-1} e^{x/y} = F(\frac{x}{y})$ is Homog

- Method: Homog D.E $y' = F(\frac{y}{x}) \rightarrow$ افهم $V = \frac{y}{x}$

اذا كانت $\frac{x}{y}$ نفس الشيء

$y = x \cdot V \rightarrow y' = x \cdot V' + V$

$V = \frac{y}{x}$ Trans From Homog to seprable

الفرق بين D.E

$xV' + V = F(V)$

$x \frac{dV}{dx} + V = F(V)$

$\frac{x}{dx} = \frac{F(V)-V}{dV}$

$\rightarrow$ sep

وكتبان افهم بالقلب

Ex: solve $x^2 y' = x^2 + xy + y^2 \rightarrow$ افهم

$y' = 1 + \frac{y}{x} + (\frac{y}{x})^2$

افهم $y' = F(\frac{y}{x})$ let $V = \frac{y}{x} \rightarrow y' = xV' + V$

افهم $xV' + V = 1 + V + V^2$

$\frac{x}{dx} = \frac{1+V^2}{dV} \rightarrow$ sep

$\int \frac{1}{x} dx = \int \frac{1}{1+V^2} dV$

$\ln(x) + C = \tan^{-1} V$

$\tan(\ln(x) + C) = \tan^{-1} \frac{y}{x}$

$y = x \tan(\ln(x) + C)$

Ex:- Solve $x' = \frac{x^2 + x\sqrt{t^2 + x^2}}{t^2}$ $x' \frac{dx}{dt} \rightarrow \text{dep}$
 $\frac{dx}{dt} \rightarrow \text{indep}$

وزعت القام $\frac{x^2}{t^2} + \frac{x\sqrt{t^2 + x^2}}{t^2} \rightarrow \frac{x}{t} + \frac{\sqrt{t^2 + x^2}}{t}$ $\rightarrow$ يا بذكر t كذا الخرافات t^2

$x' = \frac{x}{t} + \sqrt{1 + \left(\frac{x}{t}\right)^2}$ $\rightarrow$ يا بذكر t^2 كذا الخرافات من الخرافات Homog

افرض $u = \frac{x}{t} \rightarrow x = t \cdot u \rightarrow x' = t \cdot u' + u$

$t \cdot u' + u = \frac{t \cdot u}{t} + \sqrt{1 + u^2}$

$\frac{t}{dt} = \frac{u + \sqrt{1 + u^2}}{du} \rightarrow \int \frac{1}{t} dt = \int \frac{1}{u + \sqrt{1 + u^2}} du$

$(\ln t + C + \sinh^{-1}(u)) \sinh$

$t \sinh(\ln t + C) = \frac{x}{t}$

* Form $y' = G(ax + by)$ $a, b \neq 0 \in \mathbb{R}$ G is any Function

Ex [1] $\sin(2x + 3y)$

Form $G(ax + by) \rightarrow$ where $G = \sin(u)$ $u(x) = ax + by$

Ex [2] $y' = e^{3x + y + 5} \rightarrow e^{3x + y} e^5 \rightarrow \text{const}$ $\hat{y}$

Method: Use $u = ax + by$ Trans Form to sep

$y' = ax + by$

Solve $y' = \sin(x - y)$

افرض $u = x - y \rightarrow u' = 1 - y'$

$y' = u' + 1$

$\sin u = u' + 1$

$\frac{1}{dx} = \frac{\sin u - 1}{du}$

$\int dx = \int \frac{1}{\sin u - 1} du$
 x

$\int \frac{1}{\sin u - 1} \times \frac{\sin u + 1}{\sin u + 1}$

$\int \frac{1 + \sin u}{\cos^2 u} = \int \frac{1}{\cos^2 u} + \int \frac{\sin u}{\cos^2 u}$

$\int \sec^2 u = \int \frac{\sin u}{\cos^2 u} \frac{1}{\cos u}$

$\tan u + \sec u = x$

$\tan(x - y) + \sec(x - y) = x$

Implicit

$$\text{Ex: } y' = \frac{5\sqrt{x^2+6xy+9y^2}}{3} \quad y(0) = \frac{1}{3}$$

$$\frac{5\sqrt{(x+3y)^2}}{3} = \frac{1}{3}$$

$$(x+3y)^{2/5} = \frac{1}{3}$$

$$V = x+3y \rightarrow V' = 1+3y' \rightarrow y' = \frac{V'-1}{3}$$

$$(x+3y)^{2/5} = \frac{1}{3} \rightarrow \frac{2}{5} \cdot \frac{1}{3} = -\frac{1}{3} + \frac{V'}{3}$$

$$(V)^{2/5} = \frac{1}{3} + \frac{V'}{3} \rightarrow \frac{+3(V)^{2/5}}{dV} = \frac{1}{3} dx$$

$$= \int \frac{1}{V^{3/5}} dV = \int 3 dx$$

$$= \int V^{-3/5} dV = 3x$$

$$\frac{5}{3} V^{3/5} = 3x \rightarrow \frac{5}{3} \sqrt[5]{(x+3y)^3} = 3x + C$$

$$C = \frac{5}{3}$$

Ex: Find the value of m and n so that the D.E is Homog

$$(y^2 x^2 + \frac{1}{y^m}) dx = x^4 (\ln y^3 - \ln x^n) dy$$

$$\frac{y^2 x^2 + \frac{1}{y^m}}{x^4 (\ln y^3 - \ln x^n)} = \frac{dy}{dx}$$

$$\frac{x^2 (y^2 + \frac{1}{y^m})}{x^4 (\ln y^3 - \ln x^n)} = \frac{m}{n}$$

$$m = -4$$

$$n = 3$$

1] Standard Form

$a(x), b(x), c(x), g(x)$ are Fn or constant

$g(x) \neq 0$ then is Non Homog.

$$y(x) = C_1 y_1 + C_2 y_2$$
$$y'' + p(x)y' + Q(x)y = F(x)$$

- if $p(x)$, $q(x)$ and $f(x)$ cont. on interval contains x .

Then IVP has unique solu

Ex: $(1+x^2)y'' + xy' - y = \tan x$, $y(1) = 0$, $y'(1) = \pi$ Find

the Largest interval where IVP has unique solve

^٤ بدو الفتره العتملاء التي تحتوي

سہ ششویں جای الجملہ دے

X. کای

linear-IVP as

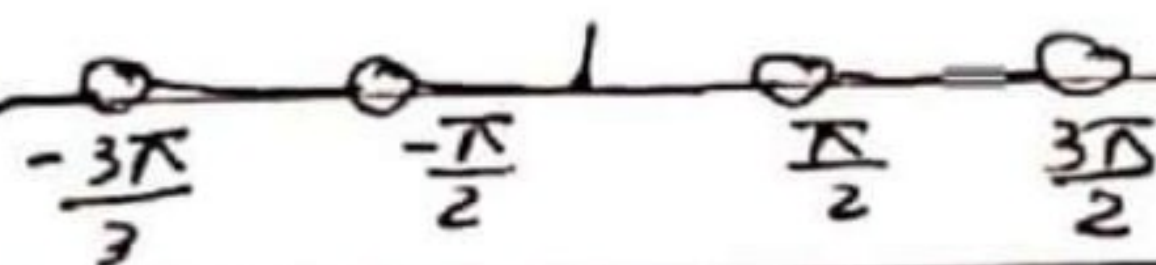
$$y'' + \underbrace{\frac{x}{1+x^2}}_{p(x)} y' - \underbrace{\frac{1}{1+x^2}}_{Q(x)} y = \underbrace{\frac{1}{1+x^2}}_{F(x)} \tan x$$

- داخلی محاسبه "lg" را و بیعت اتصال

$p(x)$ and $q(x)$ const $\rightarrow 1+x^2 \neq 0$

عشان x^2 فاصم x و x^2 فاصم x

But $f(x) = \frac{\tan x}{(1+x^2)}$ has dis cont point

~~or $x = \pm \frac{3\pi}{2}, \pm \frac{\pi}{2}, \pm \frac{5\pi}{2}$~~ 

* انا كانت x و نقطة عدم اتصال

longest interval
~~longest interval~~ $\downarrow$ also 8 different 1/1

$x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow$ largest interval

unique solve

- Defn: Two Function $f(x)$ and $g(x)$ is linearly dependent if $f(x) = \text{const}$

and IF $f(x) \neq \text{const}$ then $f(x)$ and $g(x)$ are lin-indep

$g(x)$ lin-depen: $\Rightarrow$ $f(x)$ و $g(x)$ با هم خطی هستند

lin-indep: $\Rightarrow$ $f(x)$ و $g(x)$ با هم خطی نیستند

i.e $f(x) = x^2$, $g(x) = 2x^2 \rightarrow \frac{x^2}{2x^2} = \frac{1}{2} \therefore$ lin-dep

- $f(x)$ و $g(x)$ خطی هستند

* Matrix A $n \times m$ where n : rows, m : columns

$$A_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \det(A) = |A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Defn: the wronskian of two Function $f(x)$ and $g(x)$

$$\text{is: } w[f(x), g(x)] = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = f \cdot g' - g \cdot f'$$

[1] IF $w[y_1, y_2](x) = 0$ then y_1 and y_2 are lin-dep

[2] IF $w[y_1, y_2](x) \neq 0$ then y_1 and y_2 are lin-indep

Ex: Determine the Following Functions are lin dep/indep

$$\text{III } y_1 = e^{2x}, y_2 = e^{5x} \rightarrow w(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix} = 5e^{9x} - 2e^{7x} = 3e^{7x} \neq 0$$

are lin-indep

$$\text{IV } y_1 = \sin x \cos x \rightarrow \frac{1}{2} \sin 2x$$

$$y_2 = \sin 2x \rightarrow 2 \sin x \cos x$$

$$w(y_1, y_2)$$

$$\begin{vmatrix} \sin 2x & \frac{1}{2} \sin 2x \\ 2 \cos 2x & \cos 2x \end{vmatrix}$$

$$\sin 2x \cos 2x - \frac{1}{2} \cdot 2 \sin 2x \cos 2x = 0 \rightarrow \text{dep}$$

* Fundamental Set of solution

F.S.S

بعض تحقق المعادلة

$[y_1, y_2]$ is F.S.S

For D.E $ay'' + by' + cy = 0$

(I) y_1 and y_2 are soln For D.E

غير شرطين عشوائيين

(II) y_1 and y_2 are lin-indep

F.S.S (III) y_1 and y_2 are indep

Not IF $[y_1, y_2]$ is F.S.S then general soln

is $y(x) = C_1 y_1 + C_2 y_2$

Ex: Show that $[x^3, x^4]$ F.S.S For $x^2 y'' - 6xy' + 12y = 0$ on $(0, \infty)$

(I) $\rightarrow x^3 \rightarrow y' = 3x^2, y'' = 6x$

$$x^2 \cdot 6x - 6x \cdot 3x^2 + 12 \cdot x^3 = 0 \quad \checkmark$$

(II) $x^4 \rightarrow y' = 4x^3, y'' = 12x^2$

$$x^2 \cdot 12x^2 - 6x \cdot 4x^3 + 12 \cdot x^4 = 0 \quad \checkmark$$

(III) indep

$$W[y_1, y_2] = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = x^6 \neq 0$$

عرفت ان x تتزايد من خلال الفترة $(0, \infty)$ بايجابا lin-indep

As $x \in (0, \infty)$ then $W \neq 0$

$$- y'' + P(x)y' + Q(x)y = F(x) \quad y(x_0) = y_0 \rightarrow y'(x_0) = y_1$$

I Homog Method $F(x) = 0$

II non-Homog F.S.S : $[y_1, y_2]$

- Second order Homog D.E with constant coeff.

$$\text{Forms : } ay'' + by' + cy = 0$$

$a = a$ if $a \neq 0$

where a, b, c are constant

$a \neq 0$ p.36

$$\text{Ex: I } y'' - 3y = 0 \rightarrow a=1, b=0, c=-3, \text{ Homog, second}$$

$$\text{II } y'' - 3xy = 0 \text{ Not constant coeff, Homog, second}$$

- Method : consider soln $y = e^{rx}$ where r is the root for characteristic equ.

$$- y' = r e^{rx} \quad y'' = r^2 e^{rx}$$

$$\text{In D.E : } ar^2 e^{rx} + br e^{rx} + c e^{rx} = 0$$

$$e^{rx} [ar^2 + br + c] = 0$$

$y'' = r^2$
 $y' = r$
 $y = 1$

$$\text{Char eqn : } ar^2 + br + c = 0$$

Nbt D.E $\rightarrow$ Eqn

power $\leftarrow$ "مشتق" derivative من

$$y^{(n)} \rightarrow r^n$$

$$- ar^2 + br + c = 0$$

III $\Delta = b^2 - 4ac = 0$ $\Delta > 0$ $\Delta < 0$ complex number

$$\text{Case II } \Delta > 0 \quad \Delta = b^2 - 4ac$$

we have two different roots $r_1 \neq r_2$ F.S.S = $\{e^{r_1 x}, e^{r_2 x}\}$

$$\text{general soln } y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$\text{مثال : } W[e^{r_1 x}, e^{r_2 x}] = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x} = (r_2 - r_1) e^{(r_1+r_2)x} \neq 0$$

Case II $\Delta = 0$

There are one repeated root $r_1 = r_2 = r$ F.S.S = $\{e^{rx}, x e^{rx}\}$

$$y(x) = C_1 e^{rx} + x C_2 e^{rx}$$

dep. variable x $\rightarrow$ $x e^{rx}$

Case 3 $\Delta < 0$ "complex roots"

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} \rightarrow \alpha \pm i\beta$$

$$\hat{i} = \sqrt{-1} \quad \hat{i}^2 = -1$$

Complex number $z = \alpha + i\beta$

α is real part ($\alpha = \text{Re}(z)$)

β is imaginary part ($\beta = \text{Im}(z)$)

Ex: $2 \pm 3i$
 $\alpha = 2 \quad \beta = 3$

* $\sqrt{-4} \rightarrow \sqrt{-1} \cdot \sqrt{4} = \pm 2i \rightarrow \alpha = 0 \quad \beta = 2$

* $\frac{2 + \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i \rightarrow \alpha = 1 \quad \beta = 2$

$\Delta < 0 \rightarrow$ complex root $r_{1,2} = \alpha \pm i\beta$

$$y = e^{rx} = e^{(\alpha + i\beta)x} = e^{\alpha x + i\beta x} = e^{\alpha x} \cdot e^{i\beta x}$$

use Euler Formula $e^{i\theta} = \cos \theta + i \sin \theta$

$$e^{\alpha x} [\cos(\beta x) + i \sin(\beta x)]$$

F.S.S $= e^{\alpha x} \cos(\beta x) + e^{\alpha x} \sin(\beta x) \rightarrow e^{\alpha x}$ ثابت مشترك

Ex. solve

II $y'' - 5y' + 6y = 0 \quad (r-2)(r-3)$

$\Delta = 25 - 4(1)(6) = 1 > 0 \quad r_1 = 2 \quad r_2 = 3 \quad \text{F.S.S} = \{e^{2x}, e^{3x}\}$

- general soln $y(x) = C_1 e^{2x} + C_2 e^{3x}$

II $y'' - 4y' + 4y = 0 \rightarrow 4^2 - 4(1)(4) = 0$

$(r-2)(r-2) \rightarrow r_1 = 2 \quad r_2 = 2 \quad \text{F.S.S} = \{e^{2x}, x e^{2x}\}$

- general soln $C_1 e^{2x} + C_2 x e^{2x} = y(x)$

[3] $y'' + 16y = 0 \quad \Delta < 0$

$$r^2 + 16 \rightarrow r = \frac{0 \pm \sqrt{-64}}{2} = \pm \frac{8i}{2} = \pm 4i$$

$\alpha = 0, \beta = 4$

F.S.S = $| e^{0x} \cos 4x, e^{0x} \sin 4x |$

$y(x) = C_1 \cos 4x + C_2 \sin 4x$

إذا e^{0x} متساوي 1 يعني $e^{0x} = 1$

Ex: solve $y'' - 2y' + 4y = 0$

$\Delta = 4 - 16 < 0 \quad y(0) = 0$

$r^2 - 2r + 4$

$y'(0) = 2$

$r = \frac{-(-2) \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i \quad \alpha = 1, \beta = \sqrt{3}$

F.S.S = $e^x \cos(\sqrt{3}x), e^x \sin(\sqrt{3}x)$

$y(x) = C_1 e^x \cos(\sqrt{3}x) + C_2 e^x \sin(\sqrt{3}x)$

$0 = C_1(1) + C_2(0) \rightarrow C_1 = 0$ zero

$y'(x) = C_2 e^x \cos \sqrt{3}x \cdot \sqrt{3} + C_2 x e^x \sin \sqrt{3}x$

$2 = C_2 \sqrt{3} \rightarrow C_2 = \frac{2}{\sqrt{3}}$

Ex: $r_1 = 3+i, r_2 = 3-i$

$r = 3+i$

$\hat{r} = r-1 \rightarrow \hat{r}^2 = -1$

$\hat{r} = r-3$

معادلة التفاضل من الدرجة الأولى

$-1 = (r-3)^2$

$r^2 - 6r + 9 = -1 \rightarrow r^2 - 6r + 10 \rightarrow y'' - 6y' + 10y$

* Reduction of order Method.

Let $y_1(x) \neq 0$ is a one solution for D.E

$$y'' + p(x)y' + q(x)y = 0$$

Q: How Find $y_2(x)$ y_2 and y_1 lin indep F.S. $\{y_1, y_2\}$

$$\frac{y_2}{y_1} \neq 0$$

Let $\frac{y_2(x)}{y_1(x)} = v(x)$ Consider $y_2 = v(x) y_1(x) \rightarrow$ given
to Find $v(x)$

As y_2 is solution satisfy D.E

$$y_2' = v(x) y_1' + y_1 v'$$

$$y_2'' = v y_1'' + y_1' v' + y_1 v'' + v y_1'$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

$$v y_1'' + 2y_1' v' + y_1 v'' + p(x)(v y_1' + y_1 v') + q(x)(v y_1) = 0$$

$$v [y_1'' + p(x)y_1' + q(x)y_1] + 2y_1' v' + y_1 v'' + p(x)y_1 v' = 0$$

$\rightarrow$ zero as y_1 soln

$$y_1 v'' + v' [2y_1' + p(x)y_1] = 0$$

$$v'' = - \frac{v' [2y_1' + p(x)y_1]}{y_1}$$

الخطوة الأخيرة

y_1

$$\text{الخطوة الأخيرة} \rightarrow \int \frac{v''}{v'} = \int - \frac{[2y_1' + p(x)y_1]}{y_1}$$

$$\ln v' = - \int \frac{[2y_1' + p(x)y_1]}{y_1} dx$$

Take $e^{\dots}$

$$v' = e^{-2 \ln y_1 - \int p(x) dx}$$

$$v' = \frac{1}{y_1^2} e^{-\int p(x) dx}$$

$$v = \int \frac{1}{y_1^2} e^{-\int p(x) dx} dx$$

- Reduction Formula

$$y_2 = y_1 v = y_1 \left[\int \frac{1}{y_1^2} e^{-\int p(x) dx} dx \right]$$

oo
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Subject: Ch "4"

Thu / 4 / Aug

Ex: IF $y_1(x) = e^{3x}$ is a soln to $y'' = (\frac{3}{x} + 6)y' + Q(x)y = 0 \quad x > 0$

Find the general soln.

$$p(x) = -(\frac{3}{x} + 6)$$

$$y_2 = e^{3x} \left[\int \frac{e^{+\frac{3}{x} + 6} dx}{e^{6x}} \right]$$

$$y_2 = e^{3x} \int \frac{1}{e^{6x}} x^3 e^{6x} dx$$

$$= e^{3x} \frac{x^4}{4}$$

$$\text{general soln } y(x) = C_1 e^{3x} + C_2 e^{3x} \frac{x^4}{4}$$

- Abel's Formule:

For $y'' + p(x)y' + q(x)y = 0$ if y_1, y_2 are soln

$$\text{Then, } w[y_1, y_2](x) = W(x) = C e^{-\int p(x) dx}$$

CER

$$W'(x) = C e^{-\int p(x) dx} \cdot -p(x) \rightarrow w' = w \cdot -p(x)$$

$$p(x) = -\frac{w'}{w}$$

Ex: let y_1, y_2 be soln For $x^2 y'' - 2y' + (3+x)y = 0 \quad x > 0$

Find $w[y_1, y_2](x)$

$$* 1 = y'' \text{ divide by } x^2 \rightarrow y'' - \frac{2}{x^2} y' + \left(\frac{3+x}{x^2}\right) y = 0$$

$$p(x) = -\frac{2}{x^2}$$

$$\text{by Abel's } w(x) = C e^{-\int \frac{2}{x^2} dx} = C e^{-2x^{-1}}$$

IF $w(5) = 3$ Find $w(1)$

$$3 = C e^{-\frac{2}{(5)^2}} = C e^{-1} \rightarrow C = 3e$$

$$w(5) = 3e e^{-2(5)^{-1}} \rightarrow \text{إبتعاض } 3e^{3/5}$$

* منة العبارة الجا أكثر من $w(x)$ لها أكثر من I.C بوجوب

وبتوسع ال $\frac{w(x_1)}{w(x_2)}$ ← بيطلع C مطلقا $2w(x)$ لنفس العبارة.

Ex: IF y_1 and y_2 be two soln to the D.E $y'' + p(x)y' + q(x)y = 0$ where $y_1(x) = x$ and $w(x) = x^3 e^x$ Find the general soln

$$w'(x) = x^3 e^x + 3x^2 e^x$$

$$p(x) = \frac{w'(x)}{w(x)} = \frac{x^3 e^x + 3x^2 e^x}{x^3 e^x} = -\left(1 + \frac{3}{x}\right)$$

$$y_2 = y_1 \left[\int \frac{1}{y_1^2} e^{-\int p(x) dx} dx \right]$$

$$= x \int \frac{1}{x^2} e^{5(1 + \frac{3}{x})} dx$$

$$x \int \frac{1}{x^2} e^x x^3 dx$$

$$x \int e^x x dx \rightarrow$$

$$u, du$$

$$y_2 = x(xe^x - e^x)$$

$$\begin{array}{r} x \cdot e^x \\ \downarrow + \\ 1 \cdot e^x \\ \downarrow - \\ 0 \cdot e^x \end{array}$$

$$y(x) = C_1 x + C_2 x(xe^x - e^x)$$

[3] Cauchy order $(X^n y^{(n)})$

$$ax^2 y'' + bx y' + cy = 0 \quad "x > 0" \quad a \neq 0$$

i.e

$$II \quad 3x^2 y'' + 3y = 0$$

$$a \neq 0 \quad b = 0 \quad c = 3 \rightarrow \text{Cauchy}$$

$$II \quad 4x y'' + 3y = 0$$

x^2 is not Cauchy

- Solu is $y(x) = x^r$ where r is the root For char. equ

$$y'(x) = r x^{r-1}, \quad y''(x) = r(r-1) x^{r-2}$$

As $y = x^r$ is soln For Cauchy Equ then satisfies D.E

* Cauchy substitution gives

$$ax^2 (r(r-1) x^{r-2}) + b x r x^{r-1} + c x^r = 0 \quad b r x^r, b x r x^r = x^{r-1}$$

$$ar(r-1) x^r + b r x^r + c x^r = 0$$

$$x^r [ar(r-1) + br + c] = 0 \rightarrow x^r \neq 0$$

$$\text{Char. equ} : ar(r-1) + br + c$$

$$ar^2 - ar + br + c \rightarrow ar^2 + (b-a)r + c = 0$$

[1] Case $\Delta > 0$ There is different root $r_1 \neq r_2$

$$\text{F.S.S } \{x^{r_1}, x^{r_2}\}$$

$$\text{general solu} = y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

[2] Case $\Delta = 0 \quad r_1 = r_2 = r \quad \text{F.S.S } \{x^r, x^r \ln x\}$

$$y_2 \text{ is } x^r \ln x \quad ax^2 y'' + bx y' + cy = 0 \quad y'' \text{ is } x^r \ln x$$

$$y'' + \frac{b}{ax} y' + \frac{c}{ax^2} y = 0 \rightarrow r^2 + \frac{b}{a} r + \frac{c}{a} = 0$$

$$\text{Integration} \quad r = \frac{-b+a}{2a} \rightarrow -\frac{b}{a} = ar-1, \quad y_2 = \frac{x^r}{r} \left[\int \frac{1}{x^{r+1}} dx \right] = \frac{1}{r} \int x^{-\frac{b}{a}} x^{r-1} dx$$

$$y_2 = x^r \ln x$$

[3] case $\Delta < 0$ "complex root" $r = \alpha \pm i\beta$

$$F.S.S = x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$-r_1 = \alpha + i\beta, y = x^r = x^{\alpha + i\beta} = x^\alpha x^{i\beta} = x^\alpha e^{i\beta \ln x}$$

Ex: solve the following D.E Cauchy

[1] $3x^2 y'' + 11xy' - 3y = 0 \quad x > 0$

$$a=3, b=11, c=-3$$

$$3r^2 + 8r - 3 = 0 \rightarrow \Delta = 64 - (4 \times -9) = 100 > 0$$

$$r = \frac{-8 \pm \sqrt{100}}{6}, r_1 = \frac{-8-10}{6} = -3, r_2 = \frac{-8+10}{6} = \frac{1}{3}$$

$$F.S.S | x^{1/3}, x^{-3} | \rightarrow y(x) = C_1 x^{1/3} + C_2 x^{-3}$$

[2] $x^2 y'' + 5xy' + 5y = 0$

$$a=1, b=5, c=5 \quad (5-1)$$

$$r^2 + 4r + 5 = 0 \quad \Delta = 16 - 20 = -4$$

$$r = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i \quad \alpha = -2, \beta = 1$$

[3] $t^2 y'' + ty' = 0$

$$a=1, b=1, c=0 \quad (b-a)=0$$

$$r^2 = 0 \quad F.S.S | t^0, t^0 \ln t | = 1, \ln t$$

$$y(t) = C_1 + C_2 \ln t$$

const $\rightarrow$ تابع واحد

e^{0x}, x^0 ممكن يكون

* لو بدى احد نوعى المعادلات من general soln

في اى حدينه انه المعادلات Cauchy

[Non-Homog - D.E.]

II Undetermined Coefficient Methode

$$ay'' + by' + cy = F(x) \quad \text{where } F(x) \neq 0 \rightarrow \text{Non Homog}$$

if $F(x)$ has one of the following forms:

II e^{ax+b}

II $P_n(x)$

III $\sin(ax+b)$ or $\cos(ax+b)$

general solution non-Homog $\rightarrow y(x) = y_h(x) + y_p(x)$

where $y_{h(x)}$ is homog soln if $F(x) = 0$ F.S.S $|y_1, y_2|$

$$y_{h(x)} = C_1 y_1 + C_2 y_2$$

$y_p(x)$ is the particular soln "depend on $F(x)$ and F.S.S"

Not find $y_p(x)$

II e^{ax+b}

$y_p(x)$

$$Ae^{ax+b} - A \cos b$$

II $P_n(x)$

$$ax^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

III $\sin(ax+b)$ or $\cos(ax+b)$

$$A \cos(ax+b) + B \sin(ax+b)$$

Ex: Find the form of $y_p(x)$ for the following

II $F(x) = e^{5x} \rightarrow y_p(x) = Ae^{5x}$

II $F(x) = x^4 + 2 \rightarrow y_p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$

III $F(x) = \sin 5x \rightarrow y_p(x) = A \cos 5x + B \sin 5x$

III $F(x) = \cos 3x + \sin 2x \rightarrow A \cos 3x + B \sin 3x + C \cos 2x + D \sin 2x$

لو كانو زعمت الزاوية كان الجواب $A \cos \theta + B \sin \theta$ فمجرد $A \cos + B \sin$ لا

* To Find the const. in $y_p(x)$ use $y_p(x)$ in Non-Homog D.E.

Step to solve Non-Homog

II Find F.S.S $\rightarrow |y_1, y_2|$ " $F(x) = 0$ " $y_h = C_1 y_1 + C_2 y_2$

II Find the $y_p(x)$

III general soln: $y(x) = C_1 y_1 + C_2 y_2 + y_p(x)$

Subject : Ch "2"

Mon / 8 / Aug / 2022

Ex: solve $y'' - 5y' + 4y = 8e^x$

I Non Homog $\rightarrow$ Homog $\rightarrow$ F.C.S

$$y'' - 5y' + 4y = 0 \rightarrow r^2 - 5r + 4 = 0 \rightarrow (r-1)(r-4) \rightarrow r=1, r=4$$

F.S.S $|e^x, e^{4x}|$

II $y(x) = \underbrace{c_1 e^x + c_2 e^{4x}}_{y_h(x)} + y_p(x)$

$\rightarrow = A e^x$

اذا لا e^x موجود في F.S.S بنویس

Form = $A x e^x$

في x مكان ما يغير في تكرار

III Find the const A in $y_p(x)$, y_p is soln Form

$y_p = A x e^x, y_p' = A(x+1), y_p'' = 2A e^x + A x e^x$

نضع في المعادلة $2A e^x + A x e^x - 5A e^x - 5A x e^x + 4A x e^x = 8e^x$

e^x جملان $2A - 5A = 8 \rightarrow A = -\frac{8}{3} \rightarrow y_p(x) = -\frac{8}{3} x e^x$

* general soln: $c_1 e^x + c_2 e^{4x} - \frac{8}{3} x e^x$

EX: Find the Form of $y_p(x)$ "with out const" $y'' + y = 4x + 10 \sin x$

II $y'' + y = 0 \rightarrow r^2 + 1 = 0 \rightarrow \Delta = -4 \rightarrow r = \frac{0 \pm \sqrt{-4}}{2} = \pm i$

F.S.S = $e^{\alpha} \cos \beta x, e^{\alpha} \sin \beta x = |\cos x, \sin x|$

$y_h(x) = c_1 \cos x + c_2 \sin x$

III $F_1(x) = 4x$

$F_2(x) = 10 \sin x$

$y_{1p}(x) = a_1 x + a_0$

$y_{2p}(x) = A \sin x + B \cos x$

موجود

F.S.S $\leftarrow$ مش موجود في

F.S.S مش موجود في $y_p(x)$ انا في

$y_{2p}(x) = A x \sin x + B x \cos x$

$y_p(x) = a_1 x + a_0 + A x \sin x + B x \cos x$

Ex: Find the Form of $y_p(x)$ "with out constant" $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

[I] Homog $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 \rightarrow (r-3)^2 \quad \text{F.S.S } | e^{3x}, x e^{3x} |$$

[2] $y_p(x) = a_0 + a_1 x + a_2 x^2$ $y_p(x) = A e^{3x} x^2 \rightarrow$ x^2 مرتبة x^2 مرتبة
Form of $y_p = a_0 + a_1 x + a_2 x^2 + x^2 A e^{3x}$

Ex: $y'' - 3y' - 4y = 3e^{-x} + 25 \sin x - 8e^x \cos 2x$

- Homog $\rightarrow y'' - 3y' - 4y \rightarrow r^2 - 3r - 4 \rightarrow r_1 = -1, r_2 = 4 \rightarrow \text{F.S.S } | e^{-x}, e^{4x}$

$$y_{p1} = A e^{-x} \quad y_{p2} = A \cos x + C \sin x \quad y_{p3} = e^x [D \cos 2x + E \sin 2x]$$

ما يلي راسم استجابة ثابت

$$y_p = A e^{-x} + A \cos x + C \sin x + e^x [D \cos 2x + E \sin 2x]$$

Ex: $y'' + y = 8 \sin^2 x \rightarrow$ ما يلي راسم تربيع من خلال المتطابقة

$$f(x) = 4 - 4 \cos(2x)$$

$$\frac{1}{2} = \frac{1}{2} \cos 2x$$

بموجب

- Homog $y'' + y = 0 \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow r_1 = i, r_2 = -i \rightarrow B = 1$

$$\text{F.S.S} = e^0 \cos x, e^0 \sin x \rightarrow |\cos x, \sin x|$$

$$y_{p1} = a_0 \quad y_{p2} = B \cos 2x + C \sin 2x \rightarrow y_p = a_0 + A \cos 2x + C \sin 2x$$

Ex: $y'' - y' = \sinh x \rightarrow$ ما يلي راسم توليف لا حار لهما

$$\frac{e^x - e^{-x}}{2}$$

Homog $r^2 - r = 0 \rightarrow r_1 = 0, r_2 = 1 \quad \text{F.S.S } | e^0, e^x |$

$$y_{p1} = A e^x x \quad y_{p2} = C e^{-x} \quad y_p = x A e^x + C e^{-x}$$

Ex: Given that $y_p(x) = x^2$ is a particular soln to $y'' - y = 2 - x^2$ $y(0) = 1$ $y'(0) = 0$

[I] Find the general soln.

$$\rightarrow \frac{1}{2} e^{-x} + \frac{1}{2} e^x + x^2 = \sinh + x^2$$

$$y'' - y \rightarrow r^2 = 1 \quad \text{F.S.S} = | e^{-x}, e^x |$$

$$y(x) = C_1 e^{-x} + C_2 e^x + x^2 \rightarrow y(0) \quad C_1 + C_2 = 1$$

$$y'(x) = -C_1 e^{-x} + C_2 e^x + 2x \rightarrow y'(0) \quad -C_1 + C_2 = 0$$

$$C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$$

Ex: Solve $y'' + 2y' + 2y = 5\cos x + 5\sin x$ with $y(0) = 1$, $y'(0) = 2$

$$r^2 + 2r + 2 \rightarrow \Delta = -4 \rightarrow r = \frac{-2 \pm \sqrt{4}}{2} = -1 \pm i \quad \alpha = -1, \beta = 1$$

F.S.S $|e^{-x} \cos x, e^{-x} \sin x|$

$y_p = A \cos x + B \sin x \rightarrow$ ~~$x \cos x + x \sin x$~~

عشان اوجد فلترابيت بشتغ y_p و عوف في العباره

$$y' = -A \sin x + B \cos x$$

$$y'' = -A \cos x + -B \sin x$$

$$-A \cos x - B \sin x + -2A \sin x + 2B \cos x + 2A \cos x + 2B \sin x = 5 \cos x + 5 \sin x$$

$$-A + 2B + 2A = 5 \rightarrow A + 2B = 5$$

ساري حاصلات $\cos x$

$$-B - 2A + 2B = 5 \rightarrow B - 2A = 5$$

ساري حاصلات $\sin x$

$$A = -1, B = 3$$

$$y(x) = y_h + y_p$$

$$y(x) = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x + -\cos x + 3 \sin x$$

$$y(0) = C_1 - 1 = 1 \quad C_1 = 2$$

$$y'(x) = 2(-e^{-x} \sin x + -e^{-x} \cos x) + C_2(e^{-x} \cos x + -e^{-x} \sin x) + \sin x + 3 \cos x$$

$$y'(0) = -2 + C_2 + 3 = 2 \rightarrow C_2 = 1$$

Rule: $ay'' + by' + cy = f(x)$ where $f(x) = Cx^m e^{rx}$

Then, Form

$$y(x) = x^k (A_m x^m + A_{m-1} x^{m-1} + \dots + A_1 x + A_0)$$

* S و r بتكون قديش r تكرار في r_1, r_2 و اذا ما تكررت و r مرة يكتفي $S = 0$ و r مرة يكتفي $S = 1$

اسوال كذا اوليت
الشابتر
ch "4"

Q: IF $y_1 = e^{2x}$, $y_2 = e^{-3x}$ are soln For P.E $ay'' + by' + cy = 0$. Find a, b, c

$$y_1 = e^{2x} \rightarrow r = 2 \rightarrow (r-2)$$

$$y_2 = e^{-3x} \rightarrow r = -3 \rightarrow (r+3)$$

$$\text{char eqn } (r-2)(r+3) = 0 \rightarrow r^2 + r - 6 = 0 \rightarrow y'' + y' - 6y = 0$$

$$a=1, b=1, c=-6 \Rightarrow c = -6$$

Q: $y = x^4 e^{4x}$ soln to DE $y'' + by' + cy = 0$

$$r_1 = r_2 = r = 4 \quad (r-4)^2 = r^2 - 8r + 16$$

$$a=1, b=-8, c=16$$

Not: to Find the Form of $y_{p(x)}$ if $F(x)$ is product.

Rules For DE $ay'' + by' + cy = Cx^m e^{rx}$ where $m \geq 0$ positive integer $m=0,1,2$ and r is root of char eqn.

where $m \geq 0$ positive integer $m=0,1,2$ and r is root of char eqn.

is polg. of degree m . or Then the Form of $y_{p(x)}$ is

$y_{p(x)} = x^s (A_m x^m + \dots + A_1 x + A_0) e^{rx}$ where s is the number of repetition the root r in char eqn $ar^2 + br + c = 0$

Ex: Find the Form of $y_{p(x)}$ to DE $y'' + 2y' - 3y = F(x)$ where F

[a] $F(x) = 3x e^x \rightarrow m=1, r=1, s=1$

$$\text{Homog } r^2 + 2r - 3 \rightarrow (r-1)(r+3) \rightarrow r_1 = 1, r_2 = -3$$

$$y_{p(x)} = x^1 (Ax + A_0) e^x$$

[b] $F(x) = 5x^2 e^{-2x}$

$$m=2, r=-2, s=0$$

r_1, r_2 are roots of char eqn

$$y_{p(x)} = (A_2 x^2 + A_1 x + A_0) e^{-2x}$$

Rules IF $F(x) = cx^m e^{\alpha x} \cos(\beta x)$ or $cx^m e^{\alpha x} \sin(\beta x)$ Root complex

Then Form of $y_p(x)$ is $r = \alpha \pm i\beta$ $\Delta \neq 0$

$$y_p(x) = x^s (A_m x^m + \dots + A_1 x + A_0) e^{\alpha x} \cos(\beta x) + x^s (B_m x^m + \dots + B_1 x + B_0) e^{\alpha x} \sin(\beta x)$$

where s = the number of repetition complex root $r = \alpha \pm i\beta$ in char. eqn

Ex: Find the Form of $y_p(x)$

$$\text{I} \quad y'' + 2y' - 3y = x e^{2x} \sin(3x) \rightarrow m=1, r=2 \pm 3i, s=0, \alpha=2, \beta=3$$

$$r^2 + 2r - 3 = 0 \rightarrow r_1 = 1, r_2 = -3$$

$$y_p(x) = (A_1 x + A_0) e^{2x} \cos(3x) + (B_1 x + B_0) e^{2x} \sin(3x)$$

$$\text{II} \quad y'' + 2y' + 2y = 2x e^{-x} \cos x \quad m=1, \alpha=-1, \beta=1, r=-1 \pm i, s=1$$

$$r^2 + 2r + 2 = 0 \quad \Delta = -4 \neq 0$$

$$r = -1 \pm i$$

$$y_p(x) = x (A_1 x + A_0) e^{-x} \cos x + x (B_1 x + B_0) e^{-x} \sin x$$

Ex: Solve $y'' + 2y' + y = x^2 + 1 - e^x$ with $y(0) = 0, y'(0) = 0$

$$\text{Homog } r^2 + 2r + 1 = 0 \rightarrow (r+1)^2 = 0 \rightarrow r_1 = r_2 = -1 \quad \text{F.S: } e^{-x}, e^{-x}$$

$$y_h(x) = C_1 e^{-x} + C_2 x e^{-x}$$

$$y_p(x) = a_2 x^2 + a_1 x + a_0 + B e^x$$

$$y_p'(x) = 2a_2 x + a_1 + B e^x$$

$$y_p''(x) = 2a_2 + B e^x$$

$$2a_2 + B e^x + 2(2a_2 x + a_1 + B e^x) + a_2 x^2 + a_1 x + a_0 + B e^x = x^2 + 1 - e^x$$

$$B + 2B + B = -1 \rightarrow B = -\frac{1}{4}$$

$$a_2 x^2 = 1$$

$$2a_2 x + a_1 = 0 \rightarrow a_1 = -4$$

$$2a_2 + 2a_1 + a_0 = 1 \rightarrow a_0 = 7$$

general soln

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^2 - 4x + 7 - \frac{1}{4} e^x$$

Super position principle

let $L[y] = ay'' + by' + cy$

IF y_1 be a soln to D.E $L[y_1] = f(x)$

IF y_2 be a soln to D.E $L[y_2] = g(x)$

Then soln for $L[y] = f(x) + g(x)$ is $C_1 y_1 + C_2 y_2$

Q: let $L[y] = ay'' + by' + cy$ with constant coef

IF $L[y_1] = \underbrace{2-x^2}_{f(x)}$, $L[y_2] = \underbrace{x^2+x+1}_{g(x)}$. Find soln D.E $L[y] = -5x^2-3x+1$

is $C_1 y_1 + C_2 y_2 = 2y_1 - 3y_2 \rightarrow$ الجواب

دال $\rightarrow C_1 f(x) + C_2 g(x) = -5x^2 - 3x + 1$

$C_1(2-x^2) + C_2(x^2+x+1) = -5x^2 - 3x + 1$

$-C_1 + C_2 = -5$, $|C_2 = -3| \rightarrow x^2$ عكس

$|C_1 = 2| \rightarrow$ عكس

Q: let y_1 be soln $y'' + x^2 y' + y = x^3 - 2x$, $y_2 = y'' + x^2 y' + y = 5x + 3$

Find the soln for $y'' + x^2 y' + y = 2x^2 + 11x + 4$

$C_1 y_1 + C_2 y_2 = 2x^2 + 11x + 4$

$C_1(x^3 - 2x) + C_2(5x + 3) = 2x^2 + 11x + 4$

$3C_2 = 4 \Rightarrow C_2 = \frac{4}{3} \rightarrow$ من الثابت

$C_1 = 2 \rightarrow x^2$ عكس

- Higher ODE Homog

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

* char-equa $[y^{(n)} \rightarrow r^{(n)}]$

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

II different roots $r_1 \neq r_2 \neq r_3 \dots$

$$F.S.S = e^{r_1 x}, e^{r_2 x}, e^{r_3 x}$$

III Repeated root $r_1 = r_2 = r_3$

$$F.S.S = \{e^{r_1 x}, x e^{r_1 x}, x^2 e^{r_1 x}\}$$

IV complex root $r = \alpha \pm i\beta$

$$F.S.S = e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x$$

Ex: Find general soln

$$I) y''' + 3y'' - 10y' = 0$$

$$r^3 + 3r^2 - 10r = 0$$

$$r(r^2 + 3r - 10) = 0 \quad r_1 = 0, r_2 = 2, r_3 = -5$$

$$F.S.S = \{1, e^{2x}, e^{-5x}\}, y(x) = C_1 + C_2 e^{2x} + C_3 e^{-5x}$$

$$II) y^{(5)} + 2y^{(4)} + y^{(3)} = 0$$

$$r^5 + 2r^4 + r^3 = 0 \rightarrow r^3(r^2 + 2r + 1) = 0 \rightarrow r^3(r+1)^2 = 0$$

$$r^3 = 0 \rightarrow r_1 = 0, r_2 = 0, r_3 = 0$$

$$(r+1)^2 \rightarrow r_4 = -1, r_5 = -1$$

$$F.S.S = \{1, x, x^2, e^{-x}, x e^{-x}\}$$

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x} + C_5 x e^{-x}$$

$$III) y''' + 3y'' - 4y = 0$$

$$r^3 + 3r^2 - 4 = 0$$

$$r^3 + 3r^2 - 4 = 0, r=1 \text{ is Factor } (r-1)$$

$$(r-1)(r^2 + 4r + 4) = 0 \rightarrow (r-1)(r+2)^2 = 0$$

$$r_1 = 1, r_2 = r_3 = -2$$

$$F.S.S = \{e^x, e^{-2x}, x e^{-2x}\}$$

$$\begin{array}{r} r^2 + 4r + 4 \\ r-1 \overline{) r^3 + 3r^2 - 4} \\ \underline{-r^3 - r^2} \\ 4r^2 - 4 \\ \underline{-4r^2 - 4r} \\ 4r - 4 \\ \underline{-4r - 4} \\ 0 \end{array}$$

4 $\frac{d^4 y}{dx^4} + 2 \frac{d^2 y}{dx^2} + y = 0$
 $r^4 + 2r^2 + 1 = 0 \rightarrow (r^2 + 1)(r^2 + 1)$

$r_{1,2} = \pm i, r_{3,4} = \pm i$

F.S.S = $\{ \cos x, \sin x, x \cos x, x \sin x \}$

- Q: Find the Homog D.E with constant coeff. of at least order which has the soln $y(x) = 7x^2 + 9xe^{3x} = \sin 2x$

$x^2 \rightarrow r_1 = r_2 = r_3 = 0, r_4 = 0$

$x e^{3x} \rightarrow e^{3x}, x e^{3x} \rightarrow r_4 = r_5 = 3, (r-3)^2$

$\sin 2x \rightarrow \alpha = 0, \beta = 2, r = 0 \pm 2i \rightarrow r = 2i = r^2 = -4 \rightarrow (r^2 + 4)$

$r_{6,7} = \pm 2i$

char. eqn = $r^3 (r-3)^2 (r^2 + 4) = 0, r^3 (r^2 - 6r + 9)(r^2 + 4)$

$= y^{(8)} - 6y^{(6)} + 13y^{(5)} - 20y^{(4)} + 36y^{(3)} = 0$

- Non-Homog Higher.

Ex: Find the form of $y_p(x)$ for the following D.E

1 $y''' + y'' - 2y = x e^x$

Homog $r^3 + r^2 - 2 = 0 \rightarrow r = 1, 1 + 1 - 2 = 0 \checkmark$

Sub. ans - $(r-1)(r^2 + 2r + 2) = 0$

$r = 1, r = -2 \pm \sqrt{-4} = -1 \pm i$

F.S.S = $e^x, e^{-x} \sin x, e^{-x} \cos x$

$p(x) = (a_1 x + a_0) e^x \cdot x \rightarrow$ مشتق ما يغير في تكرار

2 $y^{(4)} + y^{(3)} = 1 + e^x$

Homog $\rightarrow r^4 + r^3 = 0 \rightarrow r^3 (r+1) = 0 \rightarrow r = 0 \rightarrow r_1 = r_2 = r_3 = 0$

$r_4 = -1$ F.S.S = $[1, x, x^2, e^{-x}]$

$y_p(x) = A x^3 + B e^x$

$\rightarrow$ مشتق ما يغير في تكرار

► Subject : CH"6"

19 / Fri / Aug

Ex : Find the general soln For $y''' + y'' = 3e^x + 4x^2$

Homog $y''' + y'' = 0 \rightarrow r^3 + r^2 = 0 \rightarrow r^2(r+1) \rightarrow r_1 = r_2 = 0 \quad r_3 = -1$

F.S.S : $1, x, e^{-x}$

Form of $y_p(x)$: $y_{p1}(x) = Ae^x$

$y_{p2}(x) = a_2x^4 + a_1x^3 + a_0x^2$

$y_p(x) = Ae^x + a_2x^4 + a_1x^3 + a_0x^2$

$y_p'(x) = Ae^x + 4a_2x^3 + 3a_1x^2 + 2a_0x$

$y_p''(x) = Ae^x + 12a_2x^2 + 6a_1x + 2a_0$

$y_p'''(x) = Ae^x + 24a_2x + 6a_1$

$y_p''' + y_p'' = 3e^x + 4x^2$

$A + A = 3 \rightarrow A = \frac{3}{2}$ - ساهي e^x جملان

$12a_2 = 4 \rightarrow a_2 = \frac{1}{3}$ - ساهي x^2 جملان

$6a_1 + 24a_2 = 0 \quad a_1 = -12$ - ساهي x جملان

$2a_0 + 6a_1 = 0$ - ساهي x^0 جملان

Q: given that $(r+2)(r^2-1)^2(r^2+1)$ is the char-ecn for Linear D.E with constant coeff. [4] = 0 Find the soln

to D.E $L[y] = e^{-2x} \cos x$

$r = -2 \pm i$ $m=0$ $S=0$

$(r^2+1)=0 \rightarrow r_{2,3} = \pm i$

$r = -2$

$(r^2-1)^2 \rightarrow r_{2,3} = 1, r_{4,5} = -1$

- Variation of parameter

$y'' + p(x)y' + q(x)y = g(x) \rightarrow$ 1. if done

- No condition on $g(x)$

- Form of y_{part} without constant

- Steps

[1] Find F.S.S $[y_1, y_2]$ $y_h(x) = c_1 y_1 + c_2 y_2$ Homog.

[2] Find $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

[3] Form of $y_{part} = U_1 y_1 + U_2 y_2$

where $U_1 = - \int \frac{y_2 g(x)}{W} dx$ $U_2 = \int \frac{y_1 g(x)}{W} dx$

Ex: solve $y'' - 2y' + y = \frac{e^x}{1+x^2}$ with $y(0) = 1$ $y'(0) = -1$

[1] Homog $r^2 - 2r + 1 \rightarrow (r-1)^2 \rightarrow r_1 = r_2 = 1 \rightarrow$ F.S.S $= e^x, x e^x$

$y_h = c_1 e^x + c_2 x e^x$ [2] $W[e^x, x e^x] = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix} = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix} = e^{2x}$

[3] $U_1 = - \int \frac{x e^x \frac{e^x}{1+x^2}}{e^{2x}} dx = - \frac{1}{2} \int \frac{x}{1+x^2} dx = - \frac{1}{2} \ln |1+x^2|$

$U_2 = \int \frac{e^x \frac{e^x}{1+x^2}}{e^{2x}} dx = \int \frac{1}{1+x^2} dx = \tan^{-1} x$

$y_{part} = U_1 y_1 + U_2 y_2 = - \frac{1}{2} \ln |1+x^2| e^x + \tan^{-1} x \cdot x e^x$

- general soln

$y(x) = c_1 e^x + c_2 x e^x - \frac{1}{2} \ln |1+x^2| e^x + \tan^{-1} x \cdot x e^x$

$c_1 = 1, c_2 = -2$

Ex: solve $y'' + y = \tan x$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Homog $y'' + y = 0 \rightarrow (r^2 + 1) = 0 \rightarrow r = \pm i$

F.S.S = $[\cos x, \sin x]$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$u_1 = -\int \frac{\sin x \tan x}{1} dx = -\int \sin x \frac{\sin x}{\cos x} dx = -\int \frac{1 - \cos^2 x}{\cos x} dx = -\int \sec x - \cos x dx$
 $= -[\ln|\sec x + \tan x| - \sin x]$

$u_2 = \int \cos x \tan x dx = \int \sin x dx = -\cos x$

$y(x) = c_1 \cos x + c_2 \sin x + \cos x - \ln|\sec x + \tan x| + \sin x + \sin x \cos x$

Ex: solve $x^2 y'' - 2xy' + 2y = x^3 e^{-x}$ $x > 0$

Homog. Cauchy $a=1, b=2, c=2$ " $ar^2 + (b-a)r + c = 0$

char-eqn: $r^2 - 3r + 2 = 0 \rightarrow (r-1)(r-2) = 0$

F.S.S = $|x, x^2| \rightarrow y_{hom} = c_1 x + c_2 x^2$

Form of $g(x)$ by variation ($\div x$)

$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = x e^{-x}$ $g(x) = x e^{-x}$

$W[x, x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$u_1 = -\int \frac{x^2 \cdot x e^{-x}}{x^2} dx = -(x e^{-x} + e^{-x})$

$u_2 = \int \frac{x^2 \cdot x e^{-x}}{x^2} dx = -e^{-x}$

$y(x) = c_1 x + c_2 x^2 - e^{-x}(x+1)x - e^{-x} \cdot x^2$

Ex: solve $x^2 y' - 2xy = 6 \ln x$

$\div x \rightarrow x y' - 2y = \frac{6 \ln x}{x} \rightarrow$ Homog: $r^2 - r - 2 = 0$

F.S.S = x, x^{-1} $(r-2)(r+1)$

$y = \frac{2x}{x^3} y = \frac{6 \ln x}{x^3} \rightarrow g(x)$ $W(x^2, x^{-1}) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = 1 - 2 = -1$

$u_1 = \int x^{-1} \cdot \frac{6 \ln x}{x^3} dx = \int \frac{6 \ln x}{x^4} dx = -\frac{1}{3} x^{-3} (2 \ln x + 1)$

$u_2 = \int x^2 \cdot \frac{6 \ln x}{x^3} dx = 6 \ln x \cdot \ln x$

الجزء

Ex: Find the general soln For $y''' + y'' = 3e^x + 4x^2$

Homog $y''' + y'' = 0 \rightarrow r^3 + r^2 = 0 \rightarrow r^2(r+1) \rightarrow r_1 = r_2 = 0, r_3 = -1$

F.S.S = $1, x, e^{-x}$

Form of $y_p(x)$ $y_{p1}(x) = Ae^x$

$y_{p2}(x) = a_2x^4 + a_1x^3 + a_0x^2$

$y_p(x) = Ae^x + a_2x^4 + a_1x^3 + a_0x^2$

$y_p'(x) = Ae^x + 4a_2x^3 + 3a_1x^2 + 2a_0x$

$y_p''(x) = Ae^x + 12a_2x^2 + 6a_1x + 2a_0$

$y_p'''(x) = Ae^x + 24a_2x + 6a_1$

$y_p''' + y_p'' = 3e^x + 4x^2$

$A + A = 3 \rightarrow A = \frac{3}{2}$ - مساوي محالان e^x

$12a_2 = 4 \rightarrow a_2 = \frac{1}{3}$ - مساوي محالان x^2

$6a_1 + 24a_2 = 0 \rightarrow a_1 = -12$ - مساوي محالان x

$2a_0 + 6a_1 = 0$ - مساوي جوابی

Laplace Trans Form

ch 7

- Laplace

$$\mathcal{L}[F(t)] = F(s)$$

- Unit step

- IVP : by Laplace

Invers Laplace

$$\mathcal{L}^{-1}[F(s)] = F(t)$$

- Defn: If $F(t)$ is cont. on $(0, \infty)$ Laplace $F(t)$ is

$$\mathcal{L}[F(t)] = \int_0^{\infty} e^{-st} F(t) dt, \quad s > 0$$

$$= \lim_{k \rightarrow \infty} \left(\int_0^k e^{-st} F(t) dt \right) = F(s)$$

exist limit

$$F(s) = \mathcal{L}[F(t)]$$

لا Laplace تحويل
بسيط ولبها انواع اشكال

Ex: $\mathcal{L}[c] = \int_0^{\infty} e^{-st} c dt$ constant "c"

$$\lim_{k \rightarrow \infty} \int_0^k e^{-st} c dt = c \lim_{k \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_0^k$$

$$= \frac{c}{-s} \lim_{k \rightarrow \infty} (e^{-sk} - 1) \quad \text{as } s > 0 \quad \lim_{k \rightarrow \infty} e^{-sk} = 0$$

$$= \frac{c}{-s} (0 - 1) = \frac{c}{-s} = F(s)$$

Ex: $\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at} dt$ $a > s$ $a > s$ "a" constant

$$\lim_{k \rightarrow \infty} \int_0^k e^{(a-s)t} dt = \lim_{k \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^k$$

$$= \frac{1}{a-s} \lim_{k \rightarrow \infty} (e^{(a-s)k} - 1)$$

$$= \frac{1}{a-s} (0 - 1) = \frac{1}{s-a} = F(s)$$

- Note :

[1] $\mathcal{L}[F(t) \pm G(t)] = \mathcal{L}[F(t)] \pm \mathcal{L}[G(t)]$

[2] $\mathcal{L}[c F(t)] = c \mathcal{L}[F(t)]$

[3] $\lim_{s \rightarrow \infty} F(s) = 0$

"Note" Table For Laplace

جدول يستخدمه في تعريف إيفر الجداول

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
[1] c	c/s
[2] e^{at}	$1/s-a$
[3] $\sin(akt)$	a/s^2+a^2
[4] $\cos(bkt)$	s/s^2+b^2
[5] $\sinh(akt)$	a/s^2-a^2
[6] $\cosh(bkt)$	s/s^2-b^2
[7] t^n	$n! / s^{n+1}$

EX: Find the value of A such that

$$\mathcal{L}[F(t)] = 2A \cdot s - s^3 + 1$$

$$- \mathcal{L}[F(t)] = F(s) \lim_{s \rightarrow \infty} F(s) = 0$$

$$\lim_{s \rightarrow \infty} \frac{2A \cdot s - s^3 + 1}{s^2 - 1}$$

من قواعد لا ميلا نأخذ أكبر حدين البسط والمقام
أكبر حدين البسط والمقام

$$2A - 1 = A = +\frac{1}{2}$$

EX: Evaluate the following

[1] $\mathcal{L}[5] = 5/s$

[2] $\mathcal{L}[\sin 3t - e^{4t-5}] = \mathcal{L}[\sin 3t] - \mathcal{L}[e^{4t-5}] \rightarrow e^{4t} \cdot e^{-5}$

$$= \frac{3}{s^2+9} - e^{-5} \frac{1}{s-4}$$

[3] $\mathcal{L}[\sin(2t) \cos t] \xrightarrow{\text{ملاحظة}}$

$$\frac{1}{2} \mathcal{L}[\sin 3t + \sin t]$$

$$= \frac{1}{2} \left(\frac{3}{s^2+9} + \frac{1}{s^2+1} \right)$$

14) $\mathcal{L}[\sinh^2(5t)]$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$$\mathcal{L}\left[\frac{(e^{10t} - e^{-10t})^2}{4}\right] = \frac{1}{4}(\mathcal{L}[e^{10t} - 2e^{-10t} + e^{-10t}])$$

$$= \frac{1}{4}\left(\frac{1}{s+10} - \frac{2}{s} + \frac{1}{s+10}\right)$$

$\mathcal{L}[F(t)] = \int_0^{\infty} e^{-st} f(t) dt$

Ex: Evaluate

1) $\int_0^{\infty} e^{-st} t^3 dt = \mathcal{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$

2) $\int_0^{\infty} e^{-3t} \sin(5t) dt = \mathcal{L}[\sin(5t)] \Big|_{s=3} = \frac{5}{s^2+25} \Big|_{s=3} = \frac{5}{34}$

3) $\int_0^{\infty} 2\cos^2 t e^{-bt} dt \rightarrow F(t) = 2\cos^2 t$

$2\mathcal{L}[\cos^2 t] = 2\mathcal{L}\left[\frac{1+\cos 2t}{2}\right] = \mathcal{L}[1+\cos 2t]$

$= \frac{1}{s} + \frac{s}{s^2+4} \Big|_{s=b} = \frac{1}{b} + \frac{b}{b^2+4}$

How

4) $\int_0^{\infty} e^{-\pi t} \cos t dt = F(t) = \cos t$

$\mathcal{L}[\cos t] = \frac{s}{s^2+1} \Big|_{s=\pi} = \frac{\pi}{\pi^2+1}$

5) $\int_0^{\infty} e^{-t} e^3 dt = \mathcal{L}[e^3] = \frac{e^3}{s} \Big|_{s=1} = e^3$

Rule II) $\mathcal{L}[e^{at} F(t)] = \mathcal{L}[F(t)] \Big|_{s \rightarrow s-a}$

Ex: $\mathcal{L}[e^{\pi t} t^4] = \mathcal{L}[t^4] \Big|_{s \rightarrow s-\pi} = \frac{4!}{s^5}$

$= \frac{24}{(s-\pi)^5}$

بسیار خوب در حال درس فکرم
بسیار خوبی انا
بسیار انا کان کثیر مورد مقبول ارمی

$\mathcal{L}[e^{-3t} \cosh(2t)] = \mathcal{L}[\cosh 2t] \Big|_{s \rightarrow s+3} = \frac{s}{s^2-4} \Big|_{s=s+3} = \frac{s+3}{(s+3)^2-4}$

3) $\mathcal{L}[\sinh(3t) \cos t] = \mathcal{L}\left[\frac{(e^{3t} - e^{-3t})}{2} \cos t\right]$

$= \frac{1}{2}(\mathcal{L}[e^{3t} \cos t] - \mathcal{L}[e^{-3t} \cos t])$

$= \frac{1}{2}\left(\frac{s}{s^2+1} \Big|_{s \rightarrow s-3} - \frac{s}{s^2+1} \Big|_{s \rightarrow s+3}\right)$

Rule 121 $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} (\mathcal{L}[f(t)])$

$\mathcal{L}[t f(t)] = -\frac{d}{ds} (\mathcal{L}[f(t)])$

$\mathcal{L}[t^2 f(t)] = \frac{d^2}{ds^2} (\mathcal{L}[f(t)])$

Ex III $\mathcal{L}[t \cos t] = -\frac{d}{ds} \mathcal{L}(\cos t)$

$= -\frac{d}{ds} \left(\frac{s}{s^2+1} \right) = -1 \left(\frac{(s^2+1) - s(2s)}{(s^2+1)^2} \right) = \frac{1-s^2}{(s^2+1)^2}$

121 $\mathcal{L}[t^2 e^t]$ → بقراءه على 121 و 122 في الكتاب

$\mathcal{L}[t^2] = \frac{2!}{s^3} \Big|_{s \rightarrow s-1} = \frac{2!}{(s-1)^3}$

131 $\mathcal{L}[t e^{-t} \cos t]$ 281 $\mathcal{L}[t^n f(t) e^{at}]$ استخدم

$= -\frac{d}{ds} (\mathcal{L}[e^{-t} \cos t])$

$= -\frac{d}{ds} \left(\mathcal{L}[\cos t] \right) \Big|_{s \rightarrow s+1}$

$= -\frac{d}{ds} \left(\frac{s}{s^2+1} \right) \Big|_{s \rightarrow s+1}$

$= -\frac{d}{ds} \frac{s+1}{(s+1)^2+1} = -1 \left(\frac{2(s+1)(s+1) - ((s+1)^2+1)}{((s+1)^2+1)^2} \right)$

Inverse Laplace Transform:

If $F(t)$ is cont $(0, \infty)$

and $\mathcal{L}[F(t)] = F(s)$

$$\mathcal{L}^{-1}[F(s)] = F(t)$$

$$\mathcal{L}^{-1}[F(s-a)] = e^{at} F(t)$$

$$\mathcal{L}^{-1}[-F'(s)] = t \cdot F(t)$$

EX Determine

$$\text{III } \mathcal{L}^{-1}\left[\frac{2}{s^5} - \frac{3}{s}\right] = \mathcal{L}^{-1}\frac{2}{s^5} - \mathcal{L}^{-1}\frac{3}{s}$$

$$\frac{2}{4!} \mathcal{L}^{-1}\frac{4!}{s^5} - 3 \mathcal{L}^{-1}\frac{1}{s}$$

$$\text{Rule } \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$= \frac{1}{12} t^4 - 3 \cdot 1 = F(t)$$

$$\text{II } \mathcal{L}^{-1}\left[\frac{1}{s^2+64} - \frac{5}{s^3} + \frac{3}{s-2}\right] = \frac{1}{8} \mathcal{L}^{-1}\left[\frac{1 \cdot 8}{s^2+64}\right] - \frac{5}{2!} \mathcal{L}^{-1}\frac{2}{s^3} + 3 \mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{8} \sin 8t + \frac{5}{2} t^2 + 3e^{2t}$$

$$\text{III } \mathcal{L}^{-1}\left[\frac{3}{(s-5)^4}\right] = \mathcal{L}^{-1}\frac{3}{s^4} = \frac{3}{3!} t^3 e^{5t} \quad F(t) = \frac{1}{2} t^3$$

$$\text{IV } \mathcal{L}^{-1}\frac{s}{(s^2+4)^2} \quad F(s) = \frac{1}{s^2+4} \quad F(t) = \mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\frac{1}{s^2+4}$$

$$F(t) = \frac{1}{2} \sin 2t$$

$$F'(s) = \frac{-2s}{(s^2+4)^2} = -2 \mathcal{L}^{-1}\frac{s}{(s^2+4)^2} = t \cdot F(t)$$

$$= \frac{1}{4} t \sin 2t$$

$$\mathcal{L}^{-1}[-F'(s)] = t \cdot F(t)$$

EX: Determine

$$\text{III } \mathcal{L}^{-1}\left[\frac{5}{s^2-7s+12}\right] \quad s^2-7s+12 = (s-4)(s-3)$$

$$F(s) = \frac{5}{s^2-7s+12} = \frac{A}{(s-4)} + \frac{B}{(s-3)} \quad * 5 = A(s-3) + B(s-4)$$

$$A = 5 \quad B = -5$$

$$\mathcal{L}^{-1}\left[\frac{5}{s-4}\right] + \mathcal{L}^{-1}\left[\frac{-5}{s-3}\right] = 5e^{4t} - 5e^{3t}$$

12] $\mathcal{L}^{-1} \left[\frac{1}{s(1+s^2)} \right]$

$T(s) = \frac{A}{s} + \frac{Bs+C}{1+s^2}$
 $A + As^2 + Bs^2 + Cs = 1$

$A=1$ عكس

$A+B=0 \rightarrow B=-1$ s^2 عكس

$C=0$ s عكس

$\mathcal{L}^{-1} \left[\frac{1}{s} + \frac{-s}{1+s^2} \right] = 1 - \cos t$

13] $\mathcal{L}^{-1} \left[\frac{3}{(s^2+5)(s^2+4)} \right]$

$\mathcal{L}^{-1} F(s) = \frac{2A+B}{(s^2+5)} + \frac{2C+D}{(s^2+4)}$ H.W
 $2As^2 + 13A + 5B + 4C + 4s^2C + 5C + s^2D + 5D$

14] $\mathcal{L}^{-1} \left[\frac{s^2+9s+2}{(s+1)^2(s+3)} \right]$

Ex $\mathcal{L}^{-1} \left[\frac{3s+2}{s^2+2s+10} \right]$

$\Delta = 4 - 40 = -36 < 0$

$s^2+2s+10+1=1$

$\left(\frac{b}{2}\right)^2$ $2 \times 1 = 2$

By complete square $(s+1)^2+9$

$\mathcal{L}^{-1} \left[\frac{3s+2}{(s+1)^2+9} \right] = \mathcal{L}^{-1} \frac{3s}{(s+1)^2+9} + \mathcal{L}^{-1} \frac{2}{(s+1)^2+9}$

$3 \mathcal{L}^{-1} \frac{s+1-1}{(s+1)^2+9} \rightarrow \left(\mathcal{L}^{-1} \frac{s+1}{(s+1)^2+9} - \mathcal{L}^{-1} \frac{1}{(s+1)^2+9} \right) + \mathcal{L}^{-1} \frac{2}{(s+1)^2+9}$

$= 3 \mathcal{L}^{-1} \frac{s+1}{(s+1)^2+9} - \frac{1}{3} \mathcal{L}^{-1} \frac{1 \cdot 3}{(s+1)^2+9}$

$= 3e^{-t} \cos(3t) - \frac{1}{3} e^{-t} \sin 3t$

مثال 3* $\mathcal{L}^{-1} \left[\frac{s-a}{(s-a)^2+b^2} \right] = e^{at} \cos bt$

$\mathcal{L}^{-1} \left[\frac{b}{(s-a)^2+b^2} \right] = e^{at} \sin(bt)$

Q: IF $R(t) = \mathcal{L}^{-1} \left[\ln \left(\frac{s-1}{s+1} \right) \right]$ Find $\int_0^{\infty} e^{-xt} R(t) dt$

$\mathcal{L} F(t) = F(s)$

$s \ln \left(\frac{s-1}{s+1} \right) \Big|_{s=x} = \ln \frac{6}{8}$

$\mathcal{L} F(t) \Big|_{s=x}$

Ex $\mathcal{L}^{-1} \tan^{-1} 3s = \mathcal{L}^{-1} F(s)$

$\tan^{-1} 3s = \mathcal{L}^{-1} F(s)$

$-\frac{d}{ds} \cdot \mathcal{L}[F(s)] = \frac{d}{ds} \tan^{-1} 3s = F(s)$

$\mathcal{L}^{-1} F(s) = -\frac{d}{ds} \tan^{-1} 3s$

$\mathcal{L}^{-1} F(s) = -\left(\frac{1 \times 3}{1+9s^2} \right) = -3 \cdot \frac{1}{1+9s^2} = \frac{3}{9} \cdot \frac{1}{\frac{1}{9} + s^2}$

$\mathcal{L}^{-1} \left(\mathcal{L}[F(s)] = -\frac{1/3}{(1/3)^2 + s^2} \right) \rightarrow -\mathcal{L}[\sin(1/3)t]$

$F(s) = \sin \frac{1}{3}t$

$F(t) = \sin \left(\frac{t}{3} \right) \cdot \frac{1}{t}$

Ex: $\mathcal{L}^{-1} \left[\ln \left(\frac{s^2+1}{s-1} \right) \right] = F(x)$

$\ln \left(\frac{s^2+1}{s-1} \right) = \mathcal{L}[F(t)]$

$\ln(s^2+1) - \ln(s-1) = \mathcal{L}[F(t)]$

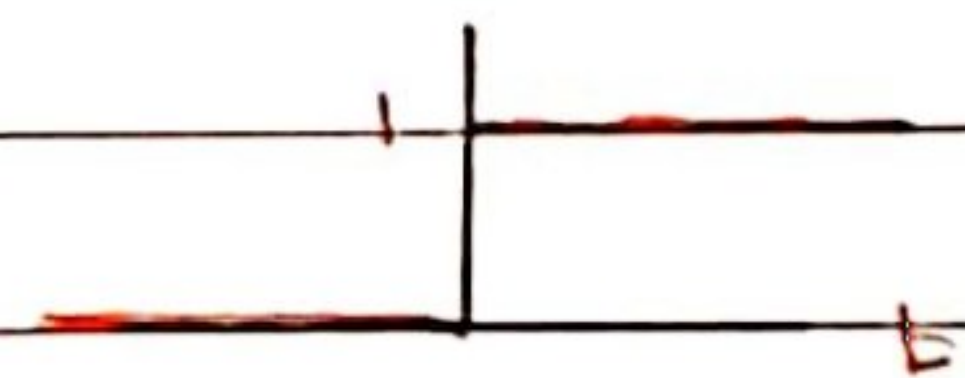
$-\frac{d}{ds} \left(\frac{2s}{s^2+1} - \frac{1}{s-1} \right) = \mathcal{L}[F(t)]$

$\mathcal{L}^{-1} \left(-2 \frac{s}{s^2+1} + \frac{1}{s-1} \right) = \mathcal{L}^{-1} \mathcal{L}[F(t)]$

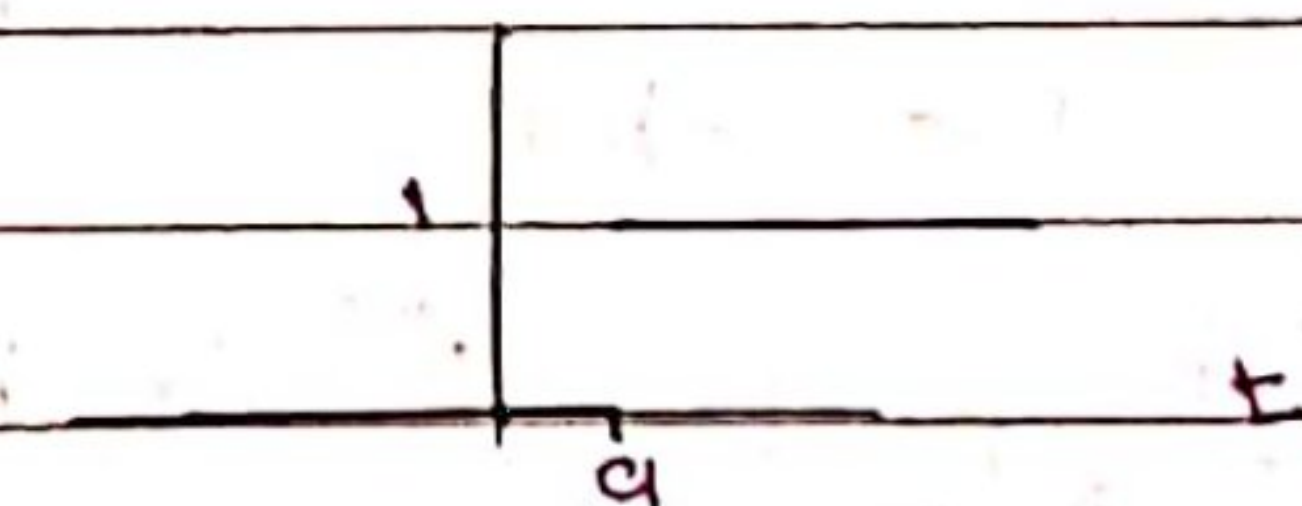
$-2 \cos t + e^t = t \cdot F(t) = \frac{-2 \cos t + e^t}{t}$

* Unit step function

[1] $U(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$



[2] $U(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$



Not

let $0 < a_1 < a_2 < \dots < a_n$ and $F(t)$ in the find by

$F(t) = F(t) \quad t < a_1$

$F(t) \quad a_1 < t < a_2$

$F(t) \quad a_{n-1} < t < a_n$

Then

$$F(t) = F_1(t) + (F_2 - F_1) U(t - a_1) + (F_3 - F_2) U(t - a_2) + \dots + (F_n - F_{n-1}) U(t - a_{n-1})$$

$$Q: \mathcal{L}[U(t-a)] = \int_0^\infty e^{-st} U(t-a) dt$$

$$= \lim_{k \rightarrow \infty} \int_0^k e^{-st} U(t-a) dt$$

$$\lim_{k \rightarrow \infty} \left(\int_0^a e^{-st} \cdot 0 dt + \int_a^k e^{-st} \cdot 1 dt \right) = \lim_{k \rightarrow \infty} \left. \frac{e^{-st}}{-s} \right|_a^k$$

$$\lim_{k \rightarrow \infty} \frac{1}{-s} (e^{-sk} - e^{-sa}) = \frac{1}{-s} (0 - e^{-as}) = \frac{e^{-as}}{s}$$

$$* \mathcal{L}^{-1} \left[\frac{e^{-as}}{s} \right] = U(t-a)$$

$$* \mathcal{L}[F(t) U(t-a)] = e^{-as} \mathcal{L}[F(t+a)]$$

EX: Find $\mathcal{L}[f(t)]$ where $f(t) = \begin{cases} 2 & t \leq 1 \\ 0 & 1 < t \leq 2 \\ e^t & t > 2 \end{cases}$

$$f(t) = 2 + (0 - 2) U(t-1) + (e^t - 0) U(t-2)$$

$$= 2 - 2U(t-1) + e^t U(t-2)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[2] - \mathcal{L}[2U(t-1)] + \mathcal{L}[e^t U(t-2)]$$

$$f(t) = \frac{2}{s} - 2 \frac{e^{-s}}{s} + \frac{e^{-2s+2}}{s-1}$$

$$e^{-2s} \mathcal{L}[F(t+2)]$$

$$e^{-2s} \mathcal{L}[e^{t+2}] = \frac{e^{-2s+2}}{s-1}$$

$$* \mathcal{L}^{-1} [e^{-as} F(s)] = U(t-a) F(t-a)$$

Subject: CH "2"

Tue/23 / Aug

$$Ex: \mathcal{L}^{-1} \frac{e^{-2s}}{s-4} \quad * a=2 \quad F(s) = \frac{1}{s-4}$$

$$F(t) = \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \left[\frac{1}{s-4} \right] = e^{4t}$$

$$U(t-2) f(t-2) = U(t-2) e^{4(t-2)}$$

$$2 e^{4t-8} U(t-2) = \begin{cases} 0 & t < 2 \\ 1 \cdot e^{4t-8} & t > 2 \end{cases}$$

$$\mathcal{L}^{-1} \left[\frac{e^{i\pi/2} s}{s^2 + 4} \right] \quad a = \frac{\pi}{2} \quad F(s) = \frac{s}{s^2 + 4}$$

$$U(t - \frac{\pi}{2}) \cos 3(t - \frac{\pi}{2})$$

$$\cos(3t - \frac{3\pi}{2})$$

$$\cos(3t) \cos(\frac{3\pi}{2}) + \sin(3t) \sin(\frac{3\pi}{2})$$

$$= \sin 3t U(t - \frac{\pi}{2})$$

$$Q: \text{IF } s > 0 \quad \int_0^{\infty} e^{-st} \ln^3 t \, dt$$

$$e^{-\ln^3} \int_0^{\infty} e^{-st} t^3 \, dt = e^{-\ln^3} \mathcal{L}[t^3] = \frac{1}{3} \frac{3!}{s^4} = \frac{2}{s^4}$$

$$* \mathcal{L}[e^{at} f(t)] = \mathcal{L}[F(s-a)] \quad s \rightarrow s-a$$

$$Q: \text{IF } \mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{2}{s}} \quad \text{Find } \mathcal{L}\left[\frac{e^{\pi t}}{\sqrt{8t}}\right]$$

$$\frac{1}{\sqrt{8}} \mathcal{L}\left[e^{\pi t} t^{-1/2}\right] = \frac{1}{\sqrt{8}} \mathcal{L}[t^{-1/2}]_{s \rightarrow s-\pi} = \frac{1}{\sqrt{8}} \sqrt{\frac{2}{s-\pi}}$$

Solving IUP by Laplace Transform

$y(t) = 0$ is soln to D.E

$$\mathcal{L}[y(t)] = Y(s)$$

$$\mathcal{L}[y''] = s^2 Y(s) - s y(0) - y'(0)$$

$$\mathcal{L}[y] = s Y(s) - y(0)$$

Ex: $ay'' + by' + cy = g(t)$ $y(0) = a$ $y'(0) = b$

F.S.S. $(y_1, y_2) \rightarrow y_p = a_n t - a_0$ $y_h = c_1 y_1 + c_2 y_2$

Steps

[1] Take Laplace For both Sides and use IC

[2] Solve For $Y(s)$ digital equation

[3] Take Laplace Inverse to Find Soln $y(t) = \mathcal{L}^{-1}[Y(s)]$

Ex: Solve I.V.P by Laplace Transform

$$y'' - y' - 6y = 2 \quad y(0) = 2 \quad y'(0) = -1$$

$$\mathcal{L}y'' - \mathcal{L}y' - 6\mathcal{L}y = \mathcal{L}2$$

$$s^2 Y(s) - 2s + 1 - sY + 2 - 6Y = 0 \rightarrow Y(s^2 - s - 6) - 2s + 3 = 0$$

$$Y = \frac{2s - 3}{s^2 - s - 6} = \mathcal{L}^{-1} \frac{2s - 3}{s^2 - s - 6} = \frac{2s - 3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$= \frac{3}{5} \mathcal{L} \frac{1}{s-3} + \frac{7}{5} \mathcal{L} \frac{1}{s+2}$$

$$(3/5)e^{3t} + (7/5)e^{-2t}$$

$$A = 3/5 \quad B = 7/5$$

Ex [2]: solve $y'' + y = t$ $y(0) = 0$ $y'(0) = 1$

Homog $y'' + y = 0 \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i$ F.S.S. $\cos t, \sin t$

$y_p(t) = at + b \rightarrow y_p'(t) = a$ $y_p''(t) = 0$

$0 + at + b = t \rightarrow a = 1, b = 0$ $y_p(t) = t$

general soln $y(t) = c_1 \cos t + c_2 \sin t + t$

$y(0) = c_1 + 0 \cdot c_2 = 0 \rightarrow \boxed{c_1 = 0}$ $y'(0) = c_2 \cos 0 + 1, y'(0) = c_2 + 1 = 1$

$\boxed{c_2 = 0}$ $y(t) = t$

Subject: ch "7"

Thu / 25 / Aug

by Laplace: $\mathcal{L}y'' + \mathcal{L}y = \mathcal{L}t$

$$s^2 Y - sy(0) - y'(0) + Y = \frac{1}{s^2}$$

$$s^2 Y - 1 + Y = \frac{1}{s^2} \rightarrow Y(s^2 + 1) = \frac{1}{s^2} + 1$$

$$Y(s^2 + 1) = \frac{1 + s^2}{s^2} \rightarrow Y = \frac{(1 + s^2)}{s^2} = \frac{1}{s^2} + \frac{1}{s^2 + 1}$$

$$\mathcal{L}^{-1} Y = \mathcal{L}^{-1} \frac{1}{s^2}$$

$$y(t) = t$$

Ex: Using Laplace Trans Form to solve $y'' - 6y' + 5y = 3e^{2t}$ $y(0) = 2$
 $y'(0) = 3$

$$\mathcal{L}y'' - 6\mathcal{L}y' - 5\mathcal{L}y = 3\mathcal{L}e^{2t}$$

$$s^2 Y - sy(0) - y'(0) - 6(sY - y(0)) - 5Y = 3 \cdot \frac{1}{s-2}$$

$$s^2 Y - 2s - 3 - 6sY + 12 + 5Y = \frac{3}{s-2}$$

$$Y[s^2 - 6s + 5] = \frac{3}{s-2} + 2s - 9$$

$$Y = \left(\frac{3}{s-2} + \frac{2s-9}{1} \right) \cdot \frac{1}{s^2 - 6s + 5}$$

$$Y(s) = \frac{3 + (2s-9)(s-2)}{(s-2)(s^2 - 6s + 5)}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{3 + (2s-9)(s-2)}{(s-2)(s-1)(s-5)} \right]$$

Ex: $y'' + 3y' + 2y = g(t)$ $y(0) = 2$ $y'(0) = -1$

where $g(t) = \begin{cases} e^{-t} & t < 3 \\ 1 & t > 3 \end{cases}$

$g(t) = e^{-t} + (1 - e^{-t}) u(t-3) = e^{-t} + u(t-3) - e^{-t} u(t-3)$

$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{e^{-t} + u(t-3) - e^{-t} u(t-3)\}$

$s^2 Y - s y(0) - y'(0) + 3(sY - y(0)) + 2Y = \frac{1}{s} + \frac{e^{-3s}}{s} - e^{-3s} \mathcal{L}\{e^{-t}\}$

$s^2 Y - 2s + 3sY - 6 + 1 + 2Y = \frac{1}{s+1} + \frac{e^{-3s}}{s} + e^{-3s} \frac{1}{s+1}$ Rule

$Y(s) [s^2 + 3s + 2] = \dots$

- Ex. Solve $y'' + ty' - 2y = -2$ $y(0) = 0$, $y'(0) = 0$

$\mathcal{L}\{y''\} + \mathcal{L}\{ty'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{-2\}$

$s^2 Y - s y(0) - y'(0) + \frac{d}{ds} (\mathcal{L}\{y'\}) - 2Y = -\frac{2}{s}$

$s^2 Y - (sY' + Y) - 2Y = -\frac{2}{s}$

$s^2 Y - sY' - Y - 2Y = -\frac{2}{s}$

$[Y(s^2 - 3) - sY'] = -\frac{2}{s}$ this differential equation First order

$Y' + (\frac{3}{s} - s)Y = \frac{2}{s^2}$ I.F. $M(s) = e^{\int (\frac{3}{s} - s) ds} = e^{\ln s^3 - s^2/2}$

$Y(s) = \frac{1}{s^3 e^{-s^2/2}} \left[\int \frac{2}{s^2} s^3 e^{-s^2/2} ds + C \right]$

$Y(s) = \frac{1}{s^3 e^{-s^2/2}} [-2e^{-s^2/2} + C]$

$u s s^2 da = 2s ds$

$\int e^{-u/2} du = -2e^{-u/2} = -2e^{-s^2/2}$

$\lim_{s \rightarrow \infty} Y(s) = 0$

$\therefore 0 + \frac{C}{s^2 e^{-s^2/2}} = 0$

$Y(s) = -\frac{2}{s^3}$

$\mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1} \left\{ -\frac{2}{s^3} \right\} \Rightarrow y(t) = -\frac{2}{2!} \mathcal{L}^{-1} \left\{ \frac{2!}{s^3} \right\} = -t^2$

Subject: ch "2"

28/ Aug/ 2022

Using Laplace Transform to solve I.V.P

$$\mathcal{L}[6(t^2-16)u(t-4)]$$

$$6^t, e^{\ln 6^t}, e^{t \ln 6}, e^t \cdot e^{(\ln 6)t} \rightarrow a$$

$$\mathcal{L}[e^{\ln 6 \cdot t} (t^2-16)u(t-4)]$$

$$\mathcal{L}[e^{t \ln 6} t^2 - 16 e^{t \ln 6} u(t-4)]$$

$$\rightarrow \mathcal{L}[t^2-16]u(t-4) \downarrow s \rightarrow s - \ln 6$$

$$e^{-4s} \mathcal{L}[t^2-16] \downarrow s \rightarrow s - \ln 6$$

♥ ... MS az pa

Chap : Subân